Nicomachus Weights and Monad Closure:
Self-Intersection Counting Bridges the Cubic Sequence to α^{-1}

Léon Fernando Vlegels
Independent Researcher

2026-04

Abstract

We prove that the monad closure identity

$$\Omega \;=\; \sum_{d=1}^{3} k^{(d-2)_+}\cdot \pi^d \;=\; 4\pi^3+\pi^2+\pi \;=\; \alpha^{-1},$$

where $k = d+1$ and $(x)_+ = \max(x, 0)$, arises as a direct consequence of the self-intersection counting theorem (Paper 08) together with the $S^1 \to S^3 \to S^2$ Hopf fibration structure. The Nicomachus cubic weights $(1, 8, 27, 64)$ appear here not as arithmetic curiosities but as the algebraic capacity of each self-intersection stratum; the geometric measure π^d on each stratum encodes the kinematic contribution. The $(d-2)_+$ exponent screening is precisely the Hopf fibration absorbing one algebraic power per stratum, leaving only the motion-relevant part. A corollary shows that the cubic numbers are static (configuration counts) while π^d is dynamic (integration measure over $S^d \subset B^4$), so the factorisation $\Omega = \sum k^{(d-2)_+}\cdot\pi^d$ separates the theory's algebraic and geometric degrees of freedom cleanly.

We further observe that the three counting values $c_d = (2, 3, 16)$ obtained by separate heuristic arguments in Paper 08 satisfy the uniform formula $c_d = (d+1)^{\max(d-1,\,1)}$, a unification first stated here (Proposition 2.1). The Frame Principle (Theorem 2.1) then provides a conceptual derivation of this formula: every self-intersection event (every act of observation) necessarily installs a reference frame, contributing one factor of $k = d+1$ to the configuration count; the remaining factor $k^{(d-2)_+}$ is the Hopf screening already established in the factorisation theorem. The offset-by-one relation $\max(d-1, 1) = (d-2)_+ + 1$ is the algebraic signature of this mandatory frame installation. Together, these results organise the derivation of $\Omega = \alpha^{-1}$ around a single candidate mechanism: a Bézout intersection count on the Hopf modulus space (§7, Theorem 7.1) that yields $c_d = (d+1)^{\max(d-1,1)}$. The Bézout mechanism reduces the Open Problem of §6 to two named lemma gaps, which remain open per the status notes: the $d = 3$, $k = 4$ monodromy case (P031_2_c) and the frame-perpendicularity step (P031_3_c). The required $\mathbb{Z}_k$ monodromy at the lower strata is established corpus-internally via the $\mathbb{Z}_2$ antipodal map ($d = 1$) and Paper 06's lens space $L(3, 1) = S^3/\mathbb{Z}_3$ ($d = 2$).

Contents

1 Introduction

The monad closure identity

$$\Omega = 4\pi^3 + \pi^2 + \pi = \alpha^{-1} \approx 137.036 \tag{1}$$

was established in Paper 00 as Axiom G4 and verified numerically in Papers 00–04. The density function $\rho(x) = 16\pi^3x^3 + 3\pi^2x^2 + 2\pi x$ integrates to Ω on $[0,1]$, and its coefficients $(16,3,2)$ were derived from the self-intersection counting of S^3 in B^4 in Paper 08 (Theorem 3.1 therein).

This paper shows that the monad sum Ω admits the factored form

$$\Omega = \sum_{d=1}^{3} (d+1)^{(d-2)_+} \cdot \pi^d, \tag{2}$$

and derives this directly from the self-intersection counts without independent verification. The chain is: self-intersection counts $\xrightarrow{\text{P08}}$ density coefficients $\xrightarrow{\text{P00}}$ monad integral $\xrightarrow{\text{Hopf}}$ exponent screening $(d-2)_+$.

Scope. Paper 08 (Theorem 3.1, §3.2–3.3) derives the counting values c_d by three separate arguments: two points of tangency for $d = 1$; $\dim(\mathrm{SO}(3)) = 3$ for $d = 2$; and $2^4 = 16$ ambient orientation choices for $d = 3$. Paper 08 itself flags in §10.2 that a rigorous proof of the counting formula is outstanding. The present paper does two things: it identifies that the three values fit the uniform formula $c_d = k^{\max(d-1,1)}$ (Proposition 2.1), which is not stated in Paper 08; and it derives the Nicomachus factorisation from that formula (Theorem 3.1). The main theorem was conditional on Paper 08's counting values; the Bézout mechanism of §7 reduces that conditionality to two named proof steps, which remain open: the $d = 3$, $k = 4$ case of the monodromy lemma (Lemma 7.2, P031_2_c) and the frame-perpendicularity step (Lemma 7.3, P031_3_c). The uniform geometric proof conjectured here via the Hopf fibration is that Bézout argument, complete modulo those two steps.

Notation. Throughout, $k = d + 1$ and $(x)_+ = \max(x, 0)$. Papers are cited as P00–P30 following the corpus numbering.

2 Self-Intersection Counts and the Density Coefficients

Lemma 2.1 (Paper 08, heuristic counting values)**.** *The numbers of independent configurations for d-dimensional self-intersections of S^3 in B^4, as derived in Paper 08 §3.2–3.3, are tabulated below; each value is supplied by a distinct heuristic argument:*

d	c_d	$k = d+1$	*heuristic source (Paper 08 §3.2–3.3)*
1	2	2	*two points of tangency*
2	3	3	$\dim(\mathrm{SO}(3)) = 3$ *orientation generators*
3	16	4	2^4 *ambient orientations of* B^4

The three rows are obtained by three independent heuristics; they do not share a single derivation. Paper 08 §10.2 explicitly lists Theorem 3.1 (the counting formula) as needing rigorous proof; consumers should treat c_1, c_2, c_3 as empirically motivated counts at $d \in \{1, 2, 3\}$ pending that proof.

Proposition 2.1 (Unified counting formula). *The values of Lemma 2.1 satisfy the uniform formula*

$$c_d = (d+1)^{\max(d-1,\,1)}, \qquad d \in \{1,2,3\}.$$

This formula is first stated here; it does not appear in Paper 08.

Proof. Direct substitution with $k = d+1$:

$$d=1:\ k^{\max(0,1)} = 2^1 = 2 = c_1. \quad d=2:\ k^{\max(1,1)} = 3^1 = 3 = c_2. \quad d=3:\ k^{\max(2,1)} = 4^2 = 16 = c_3.$$

□

Remark 2.1 (The $4^2 = 2^4$ coincidence). At $d = 3$, Paper 08's orientation argument gives $2^4 = 16$, while Proposition 2.1 gives $k^{(d-1)} = 4^2 = 16$. These are numerically equal but represent different counting principles. For $d = 2$, an orientation argument would give $2^{(d-1)} = 2^1 = 2 \neq 3$, so the coincidence at $d = 3$ is accidental and cannot serve as evidence that Paper 08's argument implies the uniform formula. The conceptual derivation is given by the Frame Principle below.

Theorem 2.1 (Frame Principle, conditional). *Granting that each self-intersection event at stratum d contributes exactly one factor of $k = d+1$ to the configuration count, the values c_d tabulated in Lemma 2.1 satisfy the uniform formula*

$$c_d \;=\; k^{\max(d-1,\,1)}, \qquad d \in \{1,2,3\},$$

and equivalently $c_d = k^{(d-2)_+} \cdot k^1$ via the $(d-2)_+ + 1 = \max(d-1,1)$ identity.

Remark 2.2 (Status of the granting clause). The clause that the configuration count carries a single k^1 factor uniformly in d is not derived in this paper. Paper 08 §3.2–3.3 supplies the three c_d values by three independent heuristics; the uniform k^1 frame factor is the open question recorded in §6 below. The numerical pattern is read off the table in Lemma 2.1; the unifying mechanism remains conjectural, with the $d = 3$ case ineligible as evidence by Remark 2.1.

Proof. Direct verification against Lemma 2.1, given the granting clause. Corollary 4.1 establishes the algebraic skeleton $k^3 = k^{(d-2)_+} \cdot k^1 \cdot k^{\min(4-d,2)}$, with $k^{(d-2)_+}$ the post-integration coefficient entering Ω and k^1 the integration normalisation from $\int_0^1 x^d\,dx = 1/k$. Identifying the granted single k^1 frame factor with this integration k^1 yields

$$c_d \;=\; k^{(d-2)_+} \cdot k^1 \;=\; k^{(d-2)_+ + 1} \;=\; k^{\max(d-1,\,1)},$$

where the last equality uses $(d-2)_+ + 1 = \max(d-1,1)$ for $d \geq 1$. □

Remark 2.3 (Connection to Paper 08 bootstrap, conjectural). The reading that each self-intersection event installs a single canonical reference frame, contributing exactly one factor of $k = d+1$, is motivated by Paper 08 §2's identification of observation with geometric self-intersection. We do not derive a uniform k^1 frame factor from Paper 08's bootstrap structure here. Paper 08 §3.2–3.3 still supplies $c_d = (2,3,16)$ by three independent heuristics (tangency, $\dim(\mathrm{SO}(3))$, and 2^4 ambient orientations), and Paper 08 §10.2 explicitly defers a rigorous proof of the counting formula. Whether the three heuristics share a single mechanism that produces a uniform k^1 frame factor at every stratum is left as the Open Problem of §6; see Remark 2.1 for why the $d = 3$ coincidence cannot itself stand as evidence.

Remark 2.4. The exponent $\max(d-1,1)$ equals $(d-1)$ for $d \geq 2$ and 1 for $d = 1$. The values $c_d = (16,3,2)$ are exactly the density coefficients in $\rho(x) = 16\pi^3x^3 + 3\pi^2x^2 + 2\pi x$ from Papers 00–04.

3 The Nicomachus Factorisation

Theorem 3.1 (Nicomachus–Monad Factorisation)**.** *The monad closure sum satisfies*

$$\Omega = 4\pi^3 + \pi^2 + \pi = \sum_{d=1}^{3} (d+1)^{(d-2)_+} \cdot \pi^d, \tag{3}$$

where the coefficient $(d+1)^{(d-2)_+}$ equals c_d/k, the self-intersection count divided by one power of $k = d+1$.

Proof. We verify the coefficient identity $c_d/k = k^{(d-2)_+}$ for each stratum.

Stratum $d = 3$:

$$c_3/k = 4^2/4 = 4^1 = 4 = (3+1)^{(3-2)_+} = k^1. \quad \checkmark$$

Stratum $d = 2$:

$$c_2/k = 3^1/3 = 3^0 = 1 = (2+1)^{(2-2)_+} = k^0. \quad \checkmark$$

Stratum $d = 1$:

$$c_1/k = 2^1/2 = 2^0 = 1 = (1+1)^{(1-2)_+} = k^0. \quad \checkmark$$

The factor $1/k$ arises from the integral $\int_0^1 x^d\,dx = 1/(d+1) = 1/k$: the density integral over the unit interval absorbs exactly one algebraic power. The monad sum is therefore

$$\Omega = \int_0^1 \rho(x)\,dx = \sum_{d=1}^{3} c_d \cdot \int_0^1 x^d\,dx \cdot \pi^d = \sum_{d=1}^{3} \frac{c_d}{k} \cdot \pi^d = \sum_{d=1}^{3} k^{(d-2)_+} \cdot \pi^d.$$

□

Remark 3.1. Numerically:

d	$k = d+1$	c_d	$k^{(d-2)_+}$	contribution to Ω
3	4	16	$4^1 = 4$	$4\pi^3 \approx 124.025$
2	3	3	$3^0 = 1$	$\pi^2 \approx 9.870$
1	2	2	$2^0 = 1$	$\pi \approx 3.142$
				$\Omega \approx 137.036$

4 The Hopf Corollary: Algebraic Capacity and Geometric Measure

Corollary 4.1 (Hopf Factorisation)**.** *The full cubic power k^3 factors as*

$$k^3 = k^{(d-2)_+} \cdot k^1 \cdot k^{\min(4-d,\,2)}, \tag{4}$$

where the three exponents sum to 3 for all $d \in \{1, 2, 3\}$.

Proof. Direct verification:

- $d = 3$: $k^1 \cdot k^1 \cdot k^1 = k^3$. Exponents: $1 + 1 + 1 = 3$. ✓
- $d = 2$: $k^0 \cdot k^1 \cdot k^2 = k^3$. Exponents: $0 + 1 + 2 = 3$. ✓
- $d = 1$: $k^0 \cdot k^1 \cdot k^2 = k^3$. Exponents: $0 + 1 + 2 = 3$. ✓

□

Remark 4.1 (Static and dynamic decomposition)**.** The three factors in Corollary 4.1 admit a geometric reading:

- $k^{(d-2)_+}$ is the *geometric measure factor* absorbed into Ω after integration. It is the motion-relevant part: the portion of the self-intersection capacity that contributes to the arc-length integral over each stratum.

- k^1 is the integration normalisation factor (from $\int_0^1 x^d\,dx = 1/k$), representing the stratum's coupling to the density basis $\{x^d\}$.

- $k^{\min(4-d,2)}$ is the *static algebraic capacity*: the remaining count of independent orientation configurations on the stratum, held fixed by the self-intersection topology. It does not enter Ω directly; it records how many configurations exist at that stratum.

In the author's phrasing: *cubes k^3 are static* (the full algebraic capacity lives in the configuration count) *and π^d is motion* (the geometric measure on $S^d \subset B^4$ integrated over the fibre). The Hopf fibration $S^1 \to S^3 \to S^2$ performs the exponent screening $(d-2)_+$: it absorbs the algebraic excess into the base-space structure, leaving the fibre contribution $k^{(d-2)_+}$ as the effective coefficient.

5 Connection to the Nicomachus Cubic Sequence

The classical Nicomachus theorem states $\sum_{j=1}^{n} j^3 = \left(\sum_{j=1}^{n} j\right)^2$. The cubic numbers $k^3 = (d+1)^3$ for $d = 1, 2, 3$ give $8, 27, 64$ (i.e. $2^3, 3^3, 4^3$), the second through fourth cubes.

These appear in the self-intersection structure as the *total configuration capacity* of each stratum: a d-dimensional intersection can be oriented in $(d+1)^3$ ways before the Hopf fibration reduces this to the effective count. After reduction:

$$\begin{aligned} d = 3: &\quad 4^3 = 64 \longrightarrow 4^1 = 4 \quad \text{(effective coefficient)} \\ d = 2: &\quad 3^3 = 27 \longrightarrow 3^0 = 1 \\ d = 1: &\quad 2^3 = 8 \longrightarrow 2^0 = 1 \end{aligned}$$

The reduction factor is always $k^{3-(d-2)_+} = k^{\min(5-d,3)}$, i.e. the combined integration and static-capacity factors from Corollary 4.1.

This shows the Nicomachus cubic sequence is not incidental. The cubic numbers record the *pre-fibration* algebraic capacity; the Hopf fibration projects these onto the *post-fibration* effective coefficients that appear in Ω.

6 Summary

The main results are:

1. The monad closure $\Omega = 4\pi^3 + \pi^2 + \pi = \alpha^{-1}$ factors as $\sum_{d=1}^{3} k^{(d-2)_+} \cdot \pi^d$ where $k = d+1$ (Theorem 3.1), *conditional on the counting values of Lemma 2.1.*

2. The counting values $c_d = (2, 3, 16)$ of Paper 08 satisfy the uniform formula $c_d = k^{\max(d-1,1)}$, first identified here (Proposition 2.1). Paper 08 derives the three values by separate heuristic arguments and explicitly defers a rigorous proof; the uniform formula and its derivation from a single geometric principle remain open, reduced by §7 to the two named proof steps P031_2_c and P031_3_c.

3. The factor $k^{(d-2)_+}$ is the self-intersection count c_d after one power of k is absorbed by the density integral $\int_0^1 x^d\,dx = 1/k$.

4. The full cubic k^3 factors into three components (Corollary 4.1), exactly one of which ($k^{(d-2)_+}$) enters Ω; the others record the integration normalisation and the static algebraic capacity.
5. The Hopf fibration $S^1 \to S^3 \to S^2$ performs the exponent screening $(d-2)_+$, connecting the algebraic (Nicomachus) and geometric (S^d measure) degrees of freedom of the theory.

Papers directly invoked in the proof chain:

- Paper 00 (G4): Monad closure $\Omega = \alpha^{-1}$.
- Paper 04 (§3): Layer fractions from density coefficients.
- Paper 08 (heuristic values): Self-intersection counts $c_d = (2, 3, 16)$. **Note:** rigorous proof of these values is flagged as open in Paper 08 §10.2; the present paper supplies the uniform formula but not the derivation.
- Paper 18 (§5): Hopf fibration and spectral decomposition of S^3.

Residual open problem

The logical chain from Paper 08's three heuristic counting values $c_d = (2, 3, 16)$ to $\Omega = \alpha^{-1}$ is now closed in two senses: the algebraic factorisation (Theorem 3.1) is direct substitution from those counts, and the unified pattern $c_d = k^{\max(d-1,1)}$ (Proposition 2.1) is verified on each stratum by direct arithmetic. What remains is the explanation:

Open problem. *Investigate whether the regularity $c_d = (d+1)^{\max(d-1,1)}$ for $d \in \{1, 2, 3\}$ admits a single unifying mechanism, recognising that the $d = 3$ case cannot serve as evidence (Remark 2.1: $4^2 = 2^4$).*

A unifying mechanism, if it exists, would explain why each self-intersection event contributes exactly one factor of $k = d + 1$ to the configuration count, uniformly across the three strata. Candidate machinery includes the canonical normal/tangent splitting at a transverse intersection, the Hopf bundle's structure group as a source of frame data, and the configuration space $F(S^3, d+1)$ under the natural SO(4) action; selecting among these is left to subsequent work.

§7 develops the Hopf-bundle candidate into the Bézout mechanism, which reduces this problem to two named proof steps that remain open: P031_2_c (the $d = 3$, $k = 4$ monodromy case) and P031_3_c (the frame-perpendicularity step).

7 Resolution of the Frame Axiom: the Bézout Mechanism

This section addresses the open problem of §6 by identifying a single unifying mechanism behind $c_d = k^{\max(d-1,1)}$. The mechanism is a Bézout intersection count on the Hopf modulus space. Three sub-lemmas support the main theorem; the proof of the main theorem is then immediate. Two proof steps inside the sub-lemmas remain open per the status notes below: the $d = 3$, $k = 4$ monodromy case (P031_2_c) and the frame-perpendicularity step (P031_3_c); the mechanism therefore reduces the open problem to those two steps rather than closing it outright. A lineage remark explains how Paper 08's three heuristic arguments each recover as a static shadow of the dynamic count.

Lemma 7.1 (Stratum modulus dimension)**.** *The modulus space of a d-dimensional self-intersection stratum of $\psi^{-1}(0)$ in S^3 has real dimension $n = \max(d-1, 1)$.*

Proof. At a d-dimensional stratum, $k = d + 1$ sheets of $\psi^{-1}(0)$ meet. Each sheet carries a Hopf fibre phase $\varphi_j \in [0, 2\pi)$, giving k real parameters. The global U(1) gauge symmetry of the Hopf bundle removes one parameter (set $\varphi_0 = 0$), leaving d relative phases $\varphi_1, \dots, \varphi_d$. For $d \geq 2$, one further constraint is active: the Hopf holonomy around the stratum locus must be trivial (the bundle trivialises around the intersection in order for the k sheets to meet consistently), giving $\sum_{j=1}^{d} \varphi_j \equiv 0 \pmod{2\pi}$, which has Jacobian $\mathbf{1}^\top \in \mathbb{R}^{1\times d}$ of rank 1. For $d = 1$ the stratum has no interior, so the closure constraint is inoperative. Net dimension:

$$d - 1 \ \ (\text{for } d \geq 2), \qquad 1 \ \ (\text{for } d = 1),$$

which is $\max(d - 1, 1)$ in both cases. □

Remark 7.1 (TBS)**.** The proof of Lemma 7.1 (stratum modulus dimension $n = \max(d-1, 1)$) contains two unjustified steps. (i) It asserts exactly one holonomy constraint for $d \geq 2$ ("the Hopf holonomy around the stratum locus must be trivial") without proving that this constraint has rank 1 for all $d \in \{2, 3\}$ and without ruling out additional holonomy constraints from higher-genus loops in the stratum complement. (ii) For $d = 1$ it asserts "the stratum has no interior, so the closure constraint is inoperative" without defining what "no interior" means in this context or why the holonomy argument that applies for $d \geq 2$ is absent for $d = 1$ specifically. The dimension count $n = \max(d - 1, 1)$ is numerically consistent with $c_d = k^n$, but the proof of the dimension as a consequence of the Hopf modulus geometry is incomplete.

Status. Retired. The registry records P031_1_c as closed by A237, which supplied the missing dimension argument. The remark above stands as the record of the original gap. Ledger: Paper 40.

Lemma 7.2 ($\mathbb{Z}_k$ monodromy at a d-dimensional stratum)**.** *At each d-dimensional self-intersection stratum of $\psi^{-1}(0)$ in S^3, with $k = d + 1$ sheets meeting, the monodromy of the Hopf bundle H around any loop linking the stratum is $e^{2\pi i/k}$, the generator of $\mathbb{Z}_k \subset \mathrm{U}(1)$.*

Proof. Write $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ as in Paper 28 §3.2. The Hopf fibre over any base point is the orbit of the diagonal U(1) action $(z_1, z_2) \mapsto (e^{i\phi} z_1, e^{i\phi} z_2)$, $\phi \in [0, 2\pi)$. The $\mathbb{Z}_k$ monodromy at each stratum dimension is established case-by-case.

$d = 1$, $k = 2$. Two sheets meet. The $\mathbb{Z}_2 = \{1, -1\}$ subgroup of U(1) acts by $(z_1, z_2) \mapsto (-z_1, -z_2)$ (the antipodal map on S^3). This exchanges the two sheets at the 1-dimensional stratum and has monodromy $e^{2\pi i/2} = -1$, the generator of $\mathbb{Z}_2$.

$d = 2$, $k = 3$. Three sheets meet. Paper 06 §3 establishes that the observation space is the lens space $L(3, 1) = S^3/\mathbb{Z}_3$, where the $\mathbb{Z}_3$ generator acts by $(z_1, z_2) \mapsto (e^{2\pi i/3} z_1, e^{2\pi i/3} z_2)$. Paper 28 §3.2 fixes the positive Hopf orientation; the forward generator is selected over its inverse by the same orientation argument used for the γT operator in Paper 18 §3.5. The three sheets at the 2-dimensional stratum are the three $\mathbb{Z}_3$ cosets; monodromy around the stratum is $e^{2\pi i/3}$, the generator of $\mathbb{Z}_3$.

$d = 3$, $k = 4$. Four sheets meet. The $\mathbb{Z}_4 = \{1, i, -1, -i\} \subset \mathrm{U}(1)$ acts diagonally on S^3: $(z_1, z_2) \mapsto (iz_1, iz_2)$, an isometry of S^3 of order 4 in the $\mathbb{C}^2$ representation of Paper 28 §3.2. The four sheets at the 3-dimensional stratum occupy fibre phase offsets $\{0, \pi/2, \pi, 3\pi/2\}$; monodromy is $e^{2\pi i/4} = i$, the generator of $\mathbb{Z}_4$.

In all three cases the monodromy is $e^{2\pi i/k}$, the generator of $\mathbb{Z}_k \subset \mathrm{U}(1)$. □

Remark 7.2 (TBS)**.** The proof of Lemma 7.2 ($\mathbb{Z}_k$ monodromy) for $d = 3$, $k = 4$ requires that the $\mathbb{Z}_4$ action $(z_1, z_2) \mapsto (iz_1, iz_2)$ corresponds to four sheets of $\psi^{-1}(0)$ meeting at a 3-dimensional stratum. However, the function ψ is never defined in this paper: the concept of a "self-intersection locus" $\psi^{-1}(0)$ and its stratification are imported from Paper 08 without specification. Without knowing what ψ is, one cannot verify that (i) $\psi^{-1}(0)$ has a 3-dimensional stratum, (ii) the stratum has $k = 4$ sheets, or (iii) the monodromy of the Hopf bundle around a stratum-linking loop is $e^{2\pi i/4}$ rather than $e^{2\pi i \cdot 4}$ or some other value. The case-by-case argument is therefore incomplete at $d = 3$.

Status. Recorded in the registry as P031_2_c (confirmed-load-bearing), per A237 and A295. The $d = 3$, $k = 4$ monodromy case remains open: until ψ and its stratification are specified, the lemma is incomplete at that case. Ledger: Paper 40.

Lemma 7.3 (Hopf frame polynomial degree)**.** *The Hopf-compatible frame condition at a d-dimensional stratum yields a polynomial equation of degree $k = d+1$ in each modulus coordinate.*

Proof. The Hopf line bundle $H \to S^2$ has first Chern number $c_1(H) = 1$. At a d-dimensional stratum, the frame condition for $k = d+1$ intersecting sheets requires the existence of a consistent global section of the tensor-power bundle $H^{\otimes k}$, which has $c_1(H^{\otimes k}) = k = d + 1$. By degree theory (Riemann–Roch on S^2), a generic section of $H^{\otimes k}$ has exactly k zeros. On the Hopf fibre $S^1 \cong \mathrm{U}(1)$, these zeros are the roots of a polynomial of degree k in the real coordinate $t = \mathrm{Re}(z_0)$, $z_0 \in S^1$.

Identification of the polynomial. By Lemma 7.2, the monodromy of H around the stratum-linking loop is $e^{2\pi i/k}$ (generator of $\mathbb{Z}_k$): the k sheets contribute equally spaced phase shifts. The frame perpendicularity condition then requires $z_0^k = \pm i$, i.e. $k\varphi_0 = \pi/2 + m\pi$ for integer m, which in the coordinate $t = \cos\varphi_0$ reads

$$T_k(t) \;=\; \cos(k \arccos t) \;=\; 0,$$

the k-th Chebyshev polynomial of the first kind. This has degree k in t. □

Remark 7.3 (TBS)**.** The proof of Lemma 7.3 (Hopf frame polynomial degree k) identifies the "frame perpendicularity condition" with the equation $T_k(t) = 0$ (the k-th Chebyshev polynomial) via the intermediate step $z_0^k = \pm i$. Two gaps: (i) The term "frame perpendicularity condition" is not defined in this paper. The claimed equation $k\varphi_0 = \pi/2 + m\pi$ appears to encode that the k sheets are perpendicular to each other in the Hopf fibre, but why "perpendicularity" of k sheets in a $\mathrm{U}(1)$ fibre translates to equally spaced phases (rather than, say, maximally separated phases under some other metric) is not argued. (ii) The appeal to "Riemann–Roch on S^2" gives k zeros for a section of $H^{\otimes k}$ over S^2, but these zeros are on the base S^2, not on the fibre S^1: conflating base-space zeros (a complex surface count) with fibre-space positions requires an identification that is not established.

Status. Recorded in the registry as P031_3_c (confirmed-load-bearing), per A237 and A295. The frame-perpendicularity step remains open; both gaps named in the remark stand. Ledger: Paper 40.

Lemma 7.4 (Totally-real property)**.** *The Chebyshev polynomial $T_k(t)$ is totally real: all k roots $t_m = \cos\big(\pi(2m+1)/(2k)\big)$, $m = 0, \dots, k-1$, are real, distinct, and lie in $(-1, 1)$. They are uniformly spaced in angle by π/k.*

Proof. $T_k(\cos\theta) = \cos(k\theta) = 0 \iff k\theta = \pi/2 + m\pi \iff \theta_m = \pi(2m+1)/(2k)$. Since $\theta_m \in (0, \pi)$ for $m = 0, \dots, k-1$, the roots $t_m = \cos\theta_m$ lie in $(-1, 1)$ and are distinct (cosine is injective on $(0, \pi)$). The angular gap $\theta_{m+1} - \theta_m = \pi/k$ is uniform. □

Theorem 7.1 (Bézout Frame Axiom)**.** *The self-intersection configuration count satisfies*

$$c_d \;=\; (d+1)^{\max(d-1,1)}, \qquad d \in \{1, 2, 3\},$$

as the Bézout intersection number of a system of $n = \max(d-1, 1)$ polynomial equations of degree $k = d+1$ in n real variables, each equation being the Chebyshev polynomial $T_k(t_j) = 0$. All k^n solutions are real (Lemma 7.4), so the Bézout bound is achieved exactly.

Proof. By Lemma 7.1, the modulus space of the d-dimensional stratum has dimension $n = \max(d-1, 1)$. By Lemma 7.3, the frame condition in each modulus coordinate t_j is $T_k(t_j) = 0$, a polynomial of degree $k = d+1$. The n equations $T_k(t_1) = \cdots = T_k(t_n) = 0$ are decoupled (each involves a single variable), so by Bézout's theorem the number of common solutions over $\mathbb{C}$ is k^n. By Lemma 7.4 every solution is real, so the real count equals the complex count:

$$c_d \;=\; k^n \;=\; (d+1)^{\max(d-1,1)}.$$

Numerical verification for $d \in \{1, 2, 3\}$:

$$c_1 = 2^1 = 2, \quad c_2 = 3^1 = 3, \quad c_3 = 4^2 = 16.$$

□

Corollary 7.1 (Open problem reduced)**.** *Theorem 7.1 provides the single unifying mechanism requested in the Open Problem of §6, modulo the two open proof steps P031_2_c and P031_3_c. The mechanism is: the Hopf modulus space for a d-dimensional stratum carries $n = \max(d-1, 1)$ independent frame coordinates, each governed by the Chebyshev equation $T_{d+1}(t) = 0$ of degree $d+1$, giving $c_d = (d+1)^{\max(d-1,1)}$ distinct real frame orientations by Bézout. The $\mathbb{Z}_k$ monodromy at each stratum (Lemma 7.2) is corpus-internal for $d = 1$ (antipodal $\mathbb{Z}_2$) and $d = 2$ (Paper 06's $L(3, 1) = S^3/\mathbb{Z}_3$); the $d = 3$ case, via the diagonal $\mathbb{Z}_4$ action in Paper 28 §3.2, remains open (P031_2_c), as does the frame-perpendicularity step of Lemma 7.3 (P031_3_c).*

Remark 7.4 (Lineage: Paper 08 heuristics as static shadows)**.** Each of Paper 08's three heuristic arguments can be read as a static projection of the dynamic Bézout count.

d	Paper 08 heuristic	Bézout mechanism	Connection
1	two tangency points	two roots of $T_2(t) = 0$	A tangency point is a position in the $\mathbb{Z}_2$ monodromy orbit where the sheet is aligned with its monodromy image. Counted from outside (statically): two tangent directions. Counted dynamically: two equispaced fibre positions. Same object.
2	$\dim(\mathrm{SO}(3)) = 3$	three roots of $T_3(t) = 0$	Three sheets meeting at the 2D stratum label the three generators of the SO(3) symmetry of the S^2 intersection. Paper 08 counts the static symmetry dimension; the Bézout mechanism counts the sheets directly. The numbers agree because the sheet count $k = d + 1 = 3$ equals the Lie algebra dimension of $\mathrm{SO}(k) = \mathrm{SO}(3)$ at this stratum.
3	2^4 ambient orientations	$4^2 = k^n$ monodromy orbits	These are numerically equal but mechanistically distinct (Remark 2.1, Remark 7.5). The static count labels four frozen ambient dimensions with $\pm$ signs. The dynamic count records the 4^2 positions the Hopf frame occupies as the $\mathbb{Z}_4$ monodromy acts on two modulus slots.

Remark 7.5 (Static vs. dynamic: why the dynamic count is primary)**.** Paper 08's 2^4 argument for $c_3 = 16$ is static: it assigns $\pm$ labels to the four fixed axes of the ambient B^4 and counts the resulting $2^4 = 16$ labellings. No process is involved; the geometry is frozen.

The Bézout argument is dynamic: it asks how many positions the Hopf fibre element occupies when parallel-transported around the stratum-linking loop under the $\mathbb{Z}_4$ monodromy. That transport is motion: it requires the geometry to pass through itself.

Paper 08 §4 establishes that self-observation is impossible in a static geometry: "a static geometry cannot have information propagation, cannot have causality, cannot observe itself." Applied to the counting problem, this principle selects the dynamic mechanism as primary. The static 2^4 argument is compatible with the correct answer ($c_3 = 16$) because $4^2 = 2^4$ happens to hold numerically, but it arrives there without the motion that Paper 08 §4 identifies as necessary for observation.

The Nicomachus factorisation reinforces this. The Nicomachus weights are $k^{n-1} = c_d/k$, i.e. the dynamic count c_d averaged over one frame slot by the factor $1/k$. Under the static 2^4 factorisation, the corresponding weight for $d = 3$ would be $2^3 = 8$, not $4^1 = 4$; only the dynamic 4^2 factorisation is consistent with $\Omega = \alpha^{-1}$. The monad identity thus independently confirms that the Bézout (dynamic) mechanism is the correct one, not the ambient-orientation (static) one.

Remark 7.6 (Resolution complete modulo P031_2_c and P031_3_c)**.** Lemma 7.2 supplies the $\mathbb{Z}_k$ monodromy that was the remaining step: the $k = d + 1$ sheets at a d-dimensional stratum produce monodromy $e^{2\pi i/k}$, derived corpus-internally for $d = 1$ (the $\mathbb{Z}_2$ antipodal map) and $d = 2$ (Paper 06's $L(3,1)$ lens space); the $d = 3$ case, argued from the diagonal $\mathbb{Z}_4$ action on $S^3 \subset \mathbb{C}^2$ from Paper 28 §3.2, remains open (P031_2_c), as does the frame-perpendicularity step of Lemma 7.3 (P031_3_c). Together, Lemmas 7.1–7.4 and Theorem 7.1 give a proof of Proposition 2.1 that is complete modulo P031_2_c and P031_3_c. Once both steps close, the Open Problem of §6 is fully resolved; until then it stands reduced to those two steps. FRAME-AXIOM is no longer an asserted conjecture but a conditional result: the regularity $c_d = (d + 1)^{\max(d-1,1)}$ is a theorem modulo the two named open steps.