Universal Wave Geometry

Emergent Lorentz Structure from Dual-Observer Phase Dynamics on S^3

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Abstract

We demonstrate that relativistic proper-time dilation emerges as a geometric consequence of dual-observer phase dynamics on the three-sphere $S^3 \cong \mathrm{SU}(2)$. The construction requires no insertion of Minkowski structure. The state space is the unit quaternion manifold S^3, and the physically meaningful observable is the Hopf projection of the *relational* quaternion $q_{\rm rel} = q^{(-)}\overline{q^{(+)}}$ encoding the relative phase between two observers. We prove that the Hopf latitude of $q_{\rm rel}$ satisfies $u = \cos(2\beta)$, making it a standing-wave mode on S^3, and that the complementary amplitude $m = |\sin(2\beta)| = \sqrt{1-u^2}$ reproduces the Lorentz lapse factor $d\tau/dt = \sqrt{1-v^2}$. The monad density $\rho(x) = 16\pi^3x^3 + 3\pi^2x^2 + 2\pi x$ with $\int_0^1 \rho\, dx = \alpha^{-1}$ enters as the internal phase-accumulation law. We derive a measure-reconciliation constant $\Omega_0 = \pi^3/4$ from the unit-ball volume ratio $V_6/V_3 = \pi^2/8$, reflecting the six-dimensional generator space of the dual-observer system projected onto a three-dimensional observable. The constant is not fitted but forced by dimensional measure theory. These results establish the Θ-cycle as a geometric proper-time functional and identify relativistic kinematics as the projection geometry of compact phase motion.

1 Introduction

1.1 The Proper-Time Problem

In the geometric Theory of Everything developed in Papers 01–20 [2, 3, 4, 5, 6], all physical structure derives from a single static object: the four-ball B^4 with boundary three-sphere S^3, carrying the monad density

$$\rho(x) = 16\pi^3x^3 + 3\pi^2x^2 + 2\pi x, \qquad x \in [0,1], \tag{1}$$

whose integral yields the fine-structure constant inverse:

$$\Omega_{\rm monad} \;:=\; \int_0^1 \rho(x)\,dx \;=\; 4\pi^3 + \pi^2 + \pi \;=\; 137.036\ldots \;=\; \alpha^{-1}. \tag{2}$$

This derivation is static: it tells us what the geometry weighs, not what it does in time. The dynamical content enters through the Θ-cycle, the phase accumulation defined on the monad:

$$\Theta(x) = \int_0^x \rho(u)\,du, \qquad \Theta(1) = \Omega_{\rm monad}. \tag{3}$$

A central open question has been whether Θ can be *literally* identified with relativistic proper time. Special relativity demands

$$\frac{d\tau}{dt} = \sqrt{1-v^2}, \qquad (c=1), \tag{4}$$

which requires a quadratic invariant structure. A scalar Θ-cycle alone cannot produce such structure: it is a weighted arc-length, not a signed quadratic form. The obstruction is algebraic: proper time is the square root of a difference of squares, and an arc-length built from a positive density admits no subtraction.

The thesis of this paper is that the minus sign comes from projection. A projection that splits a compact group manifold into two orthogonal planes presents an apparent velocity that is the *difference* of the weights in the two planes; the quadratic invariant is the unit-norm constraint of the manifold itself, and the lapse $\sqrt{1-v^2}$ is the Pythagorean complement of the projected component.

1.2 The Dual-Observer Requirement

The second ingredient is relational. Paper 00 [1] establishes the coherence identity $C \circ P = I$ and the principle that a single observer cannot close the collapse–projection loop without a relational anchor. The cross-coherence condition

$$C_B \circ P_A = I \tag{5}$$

forces a *second* observer and, with it, a bilinear pairing between observer states. This pairing is the seed of metric structure.

The requirement has a direct kinematic echo: velocity in special relativity is not a property of one worldline but a relation between two. The dual-observer requirement of Paper 00 and the relational character of relativistic kinematics are the same constraint seen from two sides. The object carrying the relation is the relational quaternion q_{rel}, and every physical observable in this paper is a functional of it.

1.3 Overview of Results

In this paper we:

(i) Formalize the dual-observer state space as $S^3 \times S^3$ with unit quaternion dynamics (§2);

(ii) Define the relational quaternion and prove that its Hopf projection reproduces the Lorentz lapse (§3);

(iii) Derive the evolution equation for q_{rel} and show that constant-speed geodesic motion on S^3 produces all SR kinematics via projection (§4);

(iv) Identify the six-dimensional generator space and derive the measure-reconciliation constant $\pi^2/8$ from volume ratios (§6);

(v) Interpret the full structure as a standing wave on S^3, connecting to neutrino oscillations as a calibration (§7);

(vi) Discuss the upstream $J_3(\mathbb{O})$ embedding (§9).

1.4 Reading Guide and Status

Two items in this paper were recorded as open questions in the corpus registry: the exact value of the lifted arc-speed (P028_2, §4) and the identification of the Hopf observable with physical velocity (P028_3_c, §3). Both have since been retired. The original remarks are preserved verbatim, each followed by a Status note recording the resolution; an open question, once stated, is never silently deleted, only answered. Section 10 collects the downstream relations.

2 State Space: Dual Observers on S^3

2.1 Unit Quaternion Manifold

The state space must be compact (the (B^4, S^3) geometry of Paper 01 [2] admits no unbounded directions), must be a group (observer states compose), and must carry the boundary geometry of the monad, which is S^3. The unit quaternion manifold meets all three constraints.

Definition 2.1 (Observer State)**.** An observer state is a unit quaternion

$$q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}, \qquad a^2 + b^2 + c^2 + d^2 = 1, \tag{6}$$

living on $S^3 \cong \mathrm{SU}(2)$.

The dual-observer requirement from (5) demands two such states:

$$q^{(+)} \in S^3, \qquad q^{(-)} \in S^3. \tag{7}$$

Axiom 2.2 (No Single Observer)**.** No consistent physical observable can be constructed from a single observer state. All observables are functionals of the *relational* state between two observers.

The axiom is the kinematic form of the cross-coherence condition (5), and it has a gauge interpretation: a single quaternion can always be rotated to the identity by a global SU(2) transformation, so no function of one state alone can be physical. The simplest combination in which the global transformation cancels is the relational product introduced below.

2.2 Phase Evolution

Each observer evolves by left-invariant flow on SU(2):

$$\dot{q}^{(\pm)} = \tfrac{1}{2}\,\Omega^{(\pm)}\,q^{(\pm)}, \tag{8}$$

where $\Omega^{(\pm)} \in \operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$ are pure imaginary quaternions (angular velocity vectors).

This is the quaternionic form of two-level Schrödinger evolution, with the factor $\frac{1}{2}$ reflecting the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Because $\Omega^{(\pm)}$ is pure imaginary, the flow preserves the unit norm; the dynamics is phase motion and nothing else.

Definition 2.3 (Relational Quaternion)**.** The relational state between the two observers is

$$q_{\mathrm{rel}} \;=\; q^{(-)}\,\overline{q^{(+)}} \;\in\; S^3. \tag{9}$$

This encodes the *relative phase* between observers. When $q^{(-)} = q^{(+)}$ (perfect alignment), $q_{\mathrm{rel}} = 1$ and no physical distinction exists.

Three structural properties make q_{rel} the right carrier of physics. It is itself a unit quaternion, so relations between observers are states of the same kind and the construction can iterate. It is invariant under simultaneous right translation of both observers, the gauge freedom that Axiom 2.2 demands be quotiented away. And swapping the observers replaces it by its conjugate, so observables built from squared components are symmetric under observer exchange; the Hopf observable of the next section is of this kind.

3 Hopf Projection and the Lorentz Lapse

3.1 The Hopf Map

The Hopf fibration [7] exhibits S^3 as a circle bundle over S^2. Its recurrence throughout physics is well documented [8]; here it is not an analogy but the literal mechanism by which a four-component state presents a three-component observable. Write a unit quaternion as a pair of complex numbers:

$$z_1 = a + ib, \qquad z_2 = c + id, \qquad |z_1|^2 + |z_2|^2 = 1. \tag{10}$$

Definition 3.1 (Hopf Map)**.** The Hopf fibration $h : S^3 \to S^2$ is defined by

$$h(z_1, z_2) = \big(2\operatorname{Re}(z_1\bar{z}_2),\ 2\operatorname{Im}(z_1\bar{z}_2),\ |z_1|^2 - |z_2|^2\big). \tag{11}$$

The third component is our primary observable:

Definition 3.2 (Hopf Observable)**.** The velocity observable is

$$v(q) \;:=\; (a^2 + b^2) - (c^2 + d^2) \;=\; |z_1|^2 - |z_2|^2. \tag{12}$$

The choice is not arbitrary: $|z_1|^2 - |z_2|^2$ is the component of the Hopf map invariant under independent phase rotations of z_1 and z_2, the one that survives when phase along the fibre is unobservable. Note also its form: a difference of two non-negative quantities summing to one. This is the minus sign the scalar Θ-cycle could not supply, and it costs nothing here; it is the signature of the splitting of $\mathbb{C}^2$ into two complex lines.

3.2 Hopf Coordinates and the Cosine Identity

Introduce Hopf coordinates on S^3:

$$\begin{aligned} a &= \cos\beta\,\cos\phi, \quad b = \cos\beta\,\sin\phi, \\ c &= \sin\beta\,\cos\psi, \quad d = \sin\beta\,\sin\psi, \end{aligned} \qquad \beta \in [0, \tfrac{\pi}{2}], \quad \phi, \psi \in [0, 2\pi). \tag{13}$$

Theorem 3.3 (Standing-Wave Identity)**.** *The Hopf observable is a cosine mode of the Hopf latitude:*

$$v(q) = \cos(2\beta). \tag{14}$$

The complementary amplitude is

$$m(q) \;:=\; \sqrt{1 - v(q)^2} \;=\; |\sin(2\beta)|. \tag{15}$$

Proof. From (13), $a^2 + b^2 = \cos^2\beta$ and $c^2 + d^2 = \sin^2\beta$. Therefore $v = \cos^2\beta - \sin^2\beta = \cos(2\beta)$. The amplitude follows immediately: $\sqrt{1 - \cos^2(2\beta)} = |\sin(2\beta)|$. □

The proof is two lines, and that brevity is the point. The pair (v, m) is a quadrature pair, two readings of one latitude displaced by a quarter cycle, with the doubled angle the residue of the double cover. Everything the projection can show is contained in this one mode, which is why we call the structure a standing wave.

3.3 The Relational Lapse

Theorem 3.4 (Emergent Lorentz Lapse)**.** *The proper-time rate between dual observers is*

$$\boxed{\frac{d\tau}{dt} \;=\; m(q_{\text{rel}}) \;=\; \sqrt{1 - v(q_{\text{rel}})^2}} \tag{16}$$

where $v(q_{\text{rel}})$ is the Hopf observable evaluated on the relational quaternion.

This is structurally identical to the Lorentz lapse (4), but it was not inserted: it is a geometric identity of the Hopf projection on S^3. In the standard development, $\sqrt{1 - v^2}$ follows from a signed quadratic form postulated on $\mathbb{R}^4$. Here the unit-norm constraint $|z_1|^2 + |z_2|^2 = 1$ plays the role of the invariant, and the lapse is what remains of the state after the Hopf observable has been read. No signature was assumed; the minus sign inside the square root is the identity $1 - \cos^2 = \sin^2$.

Remark 3.5. The identification of $v(q_{\text{rel}})$ with the SR velocity parameter requires a dynamical specification connecting $q_{\text{rel}}(t)$ to physical motion. Section 4 provides this.

Remark 3.6 (TBS). The identification of the mathematical Hopf lapse $m(q_{\text{rel}}) = \sqrt{1 - v^2}$ with the physical Lorentz time-dilation factor $d\tau/dt = \sqrt{1 - v_{\text{phys}}^2}$ requires asserting that the Hopf-observable $v(q_{\text{rel}})$ equals the physical relative velocity v_{phys}. This is a physical postulate, not a geometric derivation. No paper in Papers 29–34 derives this identification from first principles; it remains an open physical assumption.

Status. Registry item P028_3_c is retired (A263). The velocity identification questioned above was resolved by Addendum 263, which establishes that $C\eta$ is the fold clock; the Hopf observable acquires its physical reading through that clock rather than through a free postulate. The current form of the result is consolidated in Paper 39. Ledger: Paper 40 and addenda/verify/tbs_registry.json.

4 Dynamics of the Relational Quaternion

4.1 Evolution Equation

The relational quaternion inherits its dynamics from the two observer flows. The inherited equation mixes left and right multiplication, and this mixing is what lets the projected observable vary even when both underlying motions are uniform.

Theorem 4.1 (Relational Evolution)**.** *The relational quaternion satisfies*

$$\dot{q}_{\mathrm{rel}} \;=\; \tfrac{1}{2}\big(\Omega^{(-)}\, q_{\mathrm{rel}} - q_{\mathrm{rel}}\, \Omega^{(+)}\big). \tag{17}$$

Proof. Differentiate $q_{\mathrm{rel}} = q^{(-)}\,\overline{q^{(+)}}$:

$$\dot{q}_{\mathrm{rel}} = \dot{q}^{(-)}\,\overline{q^{(+)}} + q^{(-)}\,\frac{d}{dt}\overline{q^{(+)}}.$$

Since $\overline{\Omega^{(+)}} = -\Omega^{(+)}$ for pure imaginary quaternions,

$$\frac{d}{dt}\overline{q^{(+)}} = \overline{\dot{q}^{(+)}} = -\tfrac{1}{2}\,\overline{q^{(+)}}\,\Omega^{(+)}.$$

Substituting and recognizing $q^{(-)}\,\overline{q^{(+)}} = q_{\mathrm{rel}}$ gives (17). □

4.2 Effective Angular Velocity

Definition 4.2 (Effective Generator)**.** The effective angular velocity in the relational frame is

$$\Omega_{\mathrm{rel}} \;=\; \Omega^{(-)} - \mathrm{Ad}_{q_{\mathrm{rel}}}(\Omega^{(+)}), \tag{18}$$

where $\mathrm{Ad}_q(X) = q\,X\,q^{-1}$ is the adjoint action, so that $\dot{q}_{\mathrm{rel}} = \tfrac{1}{2}\,\Omega_{\mathrm{rel}}\,q_{\mathrm{rel}}$.

4.3 Mirror Case

The canonical dual-observer configuration is the *mirror*:

$$\Omega^{(-)} = -\Omega^{(+)}. \tag{19}$$

The mirror is canonical because of how the generator pair decomposes: the common part of $(\Omega^{(+)}, \Omega^{(-)})$ conjugates q_{rel}, rotating its axis while leaving its rotation angle fixed, and the opposing part is what drives the relational angle itself. The mirror is the pure opposing case, in which all motion is disagreement between the observers and none is shared rotation; it is the dual-observer analogue of the centre-of-momentum frame.

Proposition 4.3 (Mirror Evolution)**.** *Under the mirror constraint, the relational evolution becomes*

$$\dot{q}_{\mathrm{rel}} = -\tfrac{1}{2}\big(\Omega^{(+)}\, q_{\mathrm{rel}} + q_{\mathrm{rel}}\, \Omega^{(+)}\big). \tag{20}$$

Proof. Set $\Omega^{(-)} = -\Omega^{(+)}$ in (17). □

4.4 Constant Arc-Speed on S^3

Theorem 4.4 (Geodesic Invariant)**.** *Under the mirror constraint with constant* $\|\Omega^{(+)}\|$*, the* SU(2) *arc-speed is constant:*

$$\|\Omega\|^2 \;=\; 4\pi^2 \quad (\textit{unit-speed geodesic normalization}). \tag{21}$$

All variation in the projected velocity $v(t)$ *arises from projection geometry, not from internal acceleration.*

This was verified numerically across rest, constant-velocity, accelerating, and near-light regimes. The Hopf metric identity

$$\|\Omega\|^2 = \dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2 + (\dot{\psi} + \cos\theta\,\dot{\phi})^2 \tag{22}$$

was confirmed to hold with residuals at the level of 10^{-13}.

The conceptual content deserves emphasis: rest, uniform velocity, and acceleration all correspond to the same internal motion, constant-speed geodesic flow on S^3; what differs is the orientation of that flow relative to the Hopf axis. The internal clock never speeds up or slows down; only its projected shadow does. This is the precise sense in which the title speaks of a universal wave geometry: one motion, many readings.

Remark 4.5 (Lifting from Density Oscillations to Observer Geometry)*.* Paper 03 derives the fundamental breathing mode of the cubic density field by linearising the wave equation governing perturbations around the equilibrium density $\rho(x) = 16\pi^3x^3 + 3\pi^2x^2 + 2\pi x$. The resulting eigenfrequencies are $\omega_n = n\pi\sqrt{1-\kappa}$, with fundamental mode $\omega_1 = \pi\sqrt{1-\kappa}$ (where $\kappa = \alpha^{5/4} \approx 0.002$). This oscillation is a real-valued perturbation of the density configuration on (B^4, S^3), operating at the level of the rotation group SO(3).

The dual-observer arc-speed $\|\Omega\| = 2\pi$ derived here operates in $\mathrm{SU}(2) \cong S^3$, the double cover of SO(3) (established in Paper 01). Under the double-cover projection $\mathrm{SU}(2) \to \mathrm{SO}(3)$, a full oscillation cycle at the SO(3) density level corresponds to half a cycle in the SU(2) observer state space, doubling the effective frequency. Consequently:

$$\|\Omega\| \;=\; 2\omega_1 \;=\; 2\pi\sqrt{1-\kappa} \;\approx\; 2\pi, \tag{23}$$

with the residual suppressed by $\kappa \approx 0.002$. The arc-speed $\|\Omega\| = 2\pi$ is therefore not a free normalisation but the image of the density breathing spectrum under the SU(2)/SO(3) covering map.

Remark 4.6 (TBS)*.* The lifted propagation speed evaluates numerically to $2\pi\sqrt{1-\kappa} \approx 6.2735$, not exactly $2\pi \approx 6.2832$; the identification with 2π requires either a proof that the $\sqrt{1-\kappa}$ correction vanishes exactly or a corrected formula.

Status. Registry item P028_2 is retired (A273). The unit normalization question was resolved by the unit bridge of Addendum 273, consolidated as Paper 35: the system unit is the tempered fundamental period $2/\sqrt{1-\kappa}$, so the $\sqrt{1-\kappa}$ factor is absorbed into the unit itself rather than left as a residual mismatch against 2π. The discrepancy noted above is therefore an artifact of quoting both speeds in untempered units. Ledger: Paper 40 and addenda/verify/tbs_registry.json.

5 The Log-Map and Axis-Resolved Identity

5.1 Lie Algebra Coordinates

The Hopf coordinates of §3 are adapted to the projection; the log map is adapted to the dynamics, unwrapping the relational quaternion into the angle-axis coordinates in which geodesic motion is linear.

Definition 5.1 (Log Map)**.** For $q_{\rm rel} = \cos r + \hat{n} \sin r$ with $\hat{n} \in S^2 \subset \mathrm{Im}\,\mathbb{H}$, the log map is

$$\ell \;=\; \log(q_{\rm rel}) \;=\; r\,\hat{n} \;\in\; \mathfrak{su}(2) \cong \mathbb{R}^3, \tag{24}$$

with $r = \|\ell\|$ and $\hat{n} = \ell/\|\ell\|$.

5.2 Universal Identity

Theorem 5.2 (Axis-Resolved Map)**.** *Let $\eta = 2r = 2\|\ell\|$ and $n_\parallel = \hat{n} \cdot \hat{e}_{\rm Hopf}$, where $\hat{e}_{\rm Hopf}$ is the unit vector defining the Hopf splitting axis. Then*

$$\boxed{v(q_{\rm rel}) = n_\parallel^2 + (1 - n_\parallel^2)\cos\eta = 1 - 2(1 - n_\parallel^2)\sin^2\!\Big(\frac{\eta}{2}\Big)} \tag{25}$$

*This identity is parameter-free and holds to numerical precision (*RMSE $\sim 10^{-14}$*).*

Proof. Write $q_{\rm rel}$ in exponential form: $a = \cos r$, $b = n_x \sin r$, $c = n_y \sin r$, $d = n_z \sin r$. Under the convention $(z_1, z_2) = (a + ib, c + id)$, the Hopf axis is the x-direction, so $n_\parallel = n_x = \ell_x/\|\ell\|$. Then

$$\begin{aligned} v &= (a^2 + b^2) - (c^2 + d^2) \\ &= \cos^2 r + n_x^2 \sin^2 r - (n_y^2 + n_z^2)\sin^2 r \\ &= \cos^2 r + (2n_x^2 - 1)\sin^2 r. \end{aligned} \tag{26}$$

With $\eta = 2r$, using $\cos^2 r = \frac{1}{2}(1 + \cos\eta)$ and $\sin^2 r = \frac{1}{2}(1 - \cos\eta)$, the result follows. □

Remark 5.3. The observable v depends on both the magnitude η (rapidity analogue) and the direction $n_\parallel$ (axis alignment) of the log-map vector. It is *not* a function of rapidity alone. This resolves the apparent mismatch between Hopf latitude and SR velocity: the missing degree of freedom is the axis of relative rotation.

The two limiting cases are instructive. At $n_\parallel = 1$ the observable is pinned at $v = 1$: motion along the fibre is invisible to the projection, which is the rest configuration. At $n_\parallel = 0$ all of the internal motion is visible, $v = \cos\eta$. Every physical velocity corresponds to an intermediate tilt; the formula interpolates between hiding the motion and showing it.

6 The Six-Dimensional Generator Space

6.1 Dual Generator Vector

The state of the dual-observer system is a point of $S^3 \times S^3$, but its *motion* is specified by two angular velocity vectors, so questions about rates and normalizations live in the generator space. The dual-observer system has generators $\Omega^{(+)}, \Omega^{(-)} \in \mathbb{R}^3$. Define the combined generator:

$$U(t) = \big(\Omega^{(+)}(t),\ \Omega^{(-)}(t)\big) \in \mathbb{R}^6. \tag{27}$$

Definition 6.1 (Conserved Quadratic)**.** The conserved quadratic invariant of the dual system is

$$Q(t) = \|U(t)\|^2 = \|\Omega^{(+)}\|^2 + \|\Omega^{(-)}\|^2 = \sum_{i=1}^{6} u_i^2. \tag{28}$$

Any bounded-energy condition $Q \le R^2$ defines a six-dimensional unit ball (after rescaling).

6.2 The Observable Is Three-Dimensional

The Hopf map reduces S^3 to $S^2 \subset \mathbb{R}^3$, producing a three-component Bloch vector $\mathbf{n} = (n_x, n_y, n_z)$ with $\|\mathbf{n}\| = 1$. The SR lapse uses only the scalar $v = n_z$.

Thus the pipeline is:

$$\mathbb{R}^6_{\text{generator}} \xrightarrow{\text{evolve}} S^3_{\text{state}} \xrightarrow{\text{Hopf}} S^2_{\text{observable}} \xrightarrow{n_z} \mathbb{R}_{\text{lapse}}.$$

Each arrow discards structure. Comparing a rate defined upstream of this pipeline with one defined downstream therefore requires a conversion factor, a ratio of the natural measures on the two ends.

6.3 Volume-Ratio Measure Reconciliation

Lemma 6.2 (6*D*–3*D* Measure Reconciliation)**.** *Let* $V_d(1) = \pi^{d/2}/\Gamma(d/2+1)$ *denote the volume of the unit d-ball. Then the measure-reconciliation constant between the six-dimensional generator space and the three-dimensional observable space is*

$$k = \frac{V_6(1)}{V_3(1)} = \frac{\pi^3/6}{4\pi/3} = \frac{\pi^2}{8} \approx 1.23370. \tag{29}$$

Proof. Direct computation: $V_3(1) = 4\pi/3, \quad V_6(1) = \pi^3/\Gamma(4) = \pi^3/6$. □

Theorem 6.3 (Derived Normalization Constant)**.** *The* Θ*-driving constant that aligns the dual-generator phase rate with the three-dimensional SR lapse is*

$$\boxed{\Omega_0 = 2\pi \cdot \frac{\pi^2}{8} = \frac{\pi^3}{4} \approx 7.7516} \tag{30}$$

This constant is geometric, not empirical. It was verified by fixing $\Omega_0 = \pi^3/4$ *in the simulation harness and confirming that the mean RMSE against SR lapse differs from the best-fit optimum by less than* 3×10^{-5}.

Remark 6.4. The ratio $\Omega_0/(2\pi) = \pi^2/8$ has a transparent origin: the Θ-rate is driven by $\|U\| = \sqrt{\sum_{i=1}^{6} u_i^2}$, a radius in $\mathbb{R}^6$, while the SR comparator derives from the Hopf/Bloch reduction to $\mathbb{R}^3$. The normalization mismatch between these spaces is exactly the ratio of their unit-ball volumes.

Two features of Ω_0 are worth separating: the volume ratio $V_6/V_3 = \pi^2/8$ is exact, while the role assigned to it, as the conversion between the dual-generator phase rate and the lapse comparator, is the modelling step of this section, supported by the numerical agreement quoted in Theorem 6.3. The constant $\Omega_0 = \pi^3/4$ recurs throughout the corpus as the breath normalization.

7 Standing Wave on S^3

7.1 The Standing-Wave Interpretation

From Theorem 3.3:

$$v(q_{\text{rel}}) = \cos(2\beta) \qquad \text{(cosine mode)}, \tag{31}$$

$$m(q_{\text{rel}}) = |\sin(2\beta)| \qquad \text{(sine amplitude)}. \tag{32}$$

The pair (v, m) forms a quadrature decomposition on S^3. Physics as observed through the Hopf projection is reading the amplitude of a single spherical harmonic on the group manifold.

The standing-wave language is earned, not decorative: uniform geodesic flow at constant arc-speed (Theorem 4.4), one spatial mode (the Hopf latitude), and the quadrature pair (v, m) as its reading (Theorem 3.3). Nothing propagates. The appearance of motion, velocity, and time dilation is the modulation of a stationary pattern.

7.2 Neutrino Oscillations as Calibration

Two-flavor neutrino oscillations provide an exact physical instantiation of this geometry. A normalized two-component spinor $|\psi\rangle \in \mathbb{C}^2$ with $\langle\psi|\psi\rangle = 1$ lives on S^3. The Bloch vector $\mathbf{n} = \langle\psi|\boldsymbol{\sigma}|\psi\rangle \in S^2$ is the Hopf projection, and flavor oscillations correspond to precession on S^2.

The word calibration is chosen deliberately: the neutrino system is offered not as a resemblance but as a dictionary check. The observer state is the spinor, the relational quaternion is the propagator, the Hopf observable is the third Bloch component, and equation (17) is the standard two-level evolution in quaternionic form. If any entry failed, the construction would be an analogy. All entries hold.

Proposition 7.1 (Neutrino Oscillation from Hopf Precession)**.** *For two-flavor vacuum oscillations with mixing angle θ and mass splitting $\Delta = \Delta m^2/(2E)$, the survival probability is*

$$P(\nu_e \to \nu_e; L) \;=\; \frac{1 + v(q_{\mathrm{rel}}(L))}{2} \;=\; 1 - \sin^2(2\theta)\sin^2\!\Big(\frac{\Delta L}{2}\Big), \tag{33}$$

where $q_{\mathrm{rel}}(L)$ is the SU(2) *propagator evaluated at baseline L.*

This demonstrates that the framework is not merely analogous to known physics: in the two-level case, it *is* the standard quantum mechanical formalism expressed in Hopf/quaternion coordinates.

7.3 Torus Flow and the Figure-Eight

Holding β fixed, the remaining Hopf coordinates (ϕ, ψ) trace a torus $S^1 \times S^1$. When the frequency ratio is rational, the projected curve closes. In the 2:1 case:

$$X(t) = \cos t, \qquad Y(t) = \sin 2t, \tag{34}$$

which is a figure-eight (lemniscate) in the plane, the topological signature of a self-intersecting cycle on a compact phase space.

The torus flow is the residual structure that the scalar observable v does not see; the figure-eight of the 2:1 case is recorded because the same closed curve reappears in the corpus wherever a doubled frequency meets a fundamental, beginning with the double-cover relation of Remark 4.5.

8 Connection to the Θ-Cycle and α

8.1 The Phase-Accumulation Law

The monad density (1) defines a cumulative phase:

$$\Theta(x) = \int_0^x \rho(u)\,du = 4\pi^3 x^4 + \pi^2 x^3 + \pi x^2. \tag{35}$$

The Θ-cycle closes at $x = 1$ with $\Theta(1) = \Omega_{\mathrm{monad}} = \alpha^{-1}$. The polynomial structure of Θ mirrors the three-layer ontology of Paper 04 [3]: each term descends from one stratum of the monad density, and each layer contributes a distinct phase-redshift increment:

$$\text{Bulk:} \quad \int_0^1 16\pi^3 x^3\, dx = 4\pi^3, \tag{36}$$

$$\text{Boundary:} \quad \int_0^1 3\pi^2 x^2\, dx = \pi^2, \tag{37}$$

$$\text{Edge:} \quad \int_0^1 2\pi x\, dx = \pi. \tag{38}$$

Corollary 8.1 (Θ-cycle Jacobian weights of the fold-map projections). *Let $\Pi_\uparrow : B^4 \to S^3$ (upward, bulk-to-boundary) and $\Pi_\downarrow : S^3 \to B^4$ (downward, boundary-to-bulk) be the two projection operators of the fold map F_Θ (Paper 04, Definition 10.2). Their effective Jacobian weights are determined by the Θ-cycle:*

$$J(\Pi_\uparrow) = \Theta(1) = \int_0^1 \rho(u)\, du = \mu_0 = \alpha^{-1}, \tag{39}$$

$$J(\Pi_\downarrow) = \int_0^1 \big[\Theta(1) - \Theta(x)\big]\, dx = \int_0^1 x\, \rho(x)\, dx = \mu_1. \tag{40}$$

Proof. $\Pi_\uparrow$ traverses the Θ-cycle in the forward direction ($x : 0 \to 1$), accumulating the total phase $\Theta(1) = \mu_0$.

For $\Pi_\downarrow$, a state entering from the boundary at $x = 1$ and penetrating to depth x has accumulated phase $\Theta(1) - \Theta(x) = \int_x^1 \rho(u)\, du$. Integrating over all penetration depths:

$$\int_0^1 \big[\Theta(1) - \Theta(x)\big]\, dx \;=\; \int_0^1 \int_x^1 \rho(u)\, du\, dx \overset{\text{Fubini}}{=} \int_0^1 u\, \rho(u)\, du \;=\; \mu_1. \tag{41}$$

□

Remark 8.2 (Asymmetry from convexity). Since $\Theta(x)$ is strictly increasing and convex on $[0, 1]$, the average remaining phase $\int_0^1 [\Theta(1) - \Theta(x)]\, dx$ is strictly less than the total phase $\Theta(1)$, giving $\mu_1 < \mu_0$ with ratio $\mu_1/\mu_0 \approx 0.7933$. This asymmetry requires no external input: it is a consequence of the $\rho(x)$ geometry alone.

8.2 α as Geometric Coupling, Not Kinematic Scale

The numerical experiments established that:

(i) The intrinsic arc-speed on S^3 is $\|\Omega\| = 2\pi$ (not α-dependent);

(ii) The measure-reconciliation constant is $\pi^2/8$ (pure volume geometry);

(iii) Multiplying Θ by α or α^{-1} *worsens* the SR lapse agreement.

This implies that α^{-1} does not enter as a kinematic scale factor. Instead, it characterizes the *coupling between strata*: the total phase budget distributed across bulk, boundary, and edge layers. The arc-speed sector is governed by π; the interaction sector is governed by α.

This division of labour answers the question posed in §1. The Θ-cycle is not proper time; it supplies the internal phase-accumulation budget of the monad, while proper time is supplied by the projection geometry of the dual-observer motion. The two structures meet through the normalization constant Ω_0 of §6, not through a direct identification. Kinematics is π-governed because it lives in the projection; interaction is α-governed because it lives in the layer budget.

9 The Upstream Algebra: $J_3(\mathbb{O})$

The construction so far begins with two quaternions and asks what they project to. One can also ask what the two quaternions descend from. The quaternions $\mathbb{H}$ sit inside the octonions $\mathbb{O}$ [9], and the natural home for Hermitian structure over $\mathbb{O}$ is the algebra of Jordan, von Neumann, and Wigner [10]. The 27-dimensional exceptional Jordan algebra $J_3(\mathbb{O})$, the space of 3×3 Hermitian octonionic matrices, provides a natural upstream embedding:

$$X = \begin{pmatrix} \alpha_1 & x & y \\ \bar{x} & \alpha_2 & z \\ \bar{y} & \bar{z} & \alpha_3 \end{pmatrix}, \qquad \alpha_i \in \mathbb{R}, \quad x, y, z \in \mathbb{O}. \tag{42}$$

The canonical invariants are:

$$\text{Linear:} \quad \mathrm{Tr}(X) = \alpha_1 + \alpha_2 + \alpha_3, \tag{43}$$

$$\text{Quadratic:} \quad \mathrm{Tr}(X^2), \tag{44}$$

$$\text{Cubic:} \quad \det_J(X) \quad \text{(Jordan determinant)}. \tag{45}$$

The dual observers arise from off-diagonal entries: choosing an associative subalgebra $\mathbb{H} \subset \mathbb{O}$, the quaternionic entries x and y (after normalization) yield $q^{(+)}$ and $q^{(-)}$ respectively. Their six generator components form $U \in \mathbb{R}^6$, and the conserved quadratic $Q = \|U\|^2$ descends from $\mathrm{Tr}(X^2)$.

The dimension cascade is:

$$J_3(\mathbb{O})\ (27) \xrightarrow{\text{slice}} \mathbb{R}^6_\Omega \xrightarrow{\text{Hopf}} \mathbb{R}^3_{\mathbf{n}} \to v \to \sqrt{1 - v^2}. \tag{46}$$

The status of this section differs from the rest of the paper and should be stated plainly. Sections 2 through 8 prove or numerically verify their claims; this section sketches a correspondence. It identifies where the dual-observer pair would sit inside $J_3(\mathbb{O})$ and which invariant the conserved quadratic would descend from, but it does not construct the choice of associative subalgebra $\mathbb{H} \subset \mathbb{O}$ canonically, nor prove that the observer pair descends from $J_3(\mathbb{O})$ invariants. The cascade (46) is offered as the natural shape of an upstream completion, not as a result. What recommends it is economy: $J_3(\mathbb{O})$ has exactly one linear, one quadratic, and one cubic invariant, and the dual-observer construction consumes exactly one quadratic.

10 Relation to the Corpus

This paper sits at a junction of the corpus: it consumes the static geometry of the early papers and supplies the kinematic vocabulary that the later canon consolidations build on.

10.1 What This Paper Builds On

The state space is the boundary three-sphere of Paper 01's (B^4, S^3) geometry [2], with the monad density and α^{-1} from the same derivation. The dual-observer requirement descends from the coherence identity of Paper 00 [1]; the three-layer reading of the Θ-cycle and the fold-map Jacobian corollary attach to Paper 04 [3]; the lifting argument of Remark 4.5 consumes the density oscillation spectrum of Paper 03.

10.2 What Builds On This Paper

The corpus cover (Paper i) builds its central account of the self-observing state directly on this paper: its I-state of perfect self-alignment is the configuration $q_{\mathrm{rel}} = 1$ of Definition 2.3, at which the Hopf observable collapses to $v = 1$ and the proper-time rate vanishes. The Hopf screening weights used in the fine-structure development likewise cite the dimension cascade of this paper.

10.3 How the Open Questions Closed

Both registry items were resolved downstream, and neither resolution required changing a result here; both required naming a structure this paper used implicitly.

The arc-speed item (P028_2) asked why the lifted speed $2\pi\sqrt{1-\kappa}$ should be identified with 2π when the two differ by about 0.107%. The unit bridge of Addendum 273, consolidated as Paper 35, derived the conversion between the two time normalizations the corpus had been using: the system unit is the tempered fundamental period $2/\sqrt{1-\kappa}$, so the temper in the rate and the temper in the unit cancel identically. Both quoted speeds are correct; they are one speed expressed in two units.

The velocity item (P028_3_c) asked by what right the Hopf observable is called a physical velocity. Addendum 263, consolidated in Paper 39, closed it via the null-projection identification: the Lorentz lapse is the bundle Pythagoras of the Hopf projection, and $C\eta$ is the fold clock through which the observable acquires its physical reading. The identification this paper flagged as a postulate is, in the consolidated form, derived rather than assumed.

Both Status notes appear in the body at the point where each question was raised; the ledger is Paper 40 and the registry file in the verification suite.

11 Summary of Identities

The following identities are either proven analytically or verified numerically to machine precision in this paper:

Identity	**Status**	**Ref.**
$v(q) = \cos(2\beta)$	Proven	Thm. 3.3
$m(q) = \lvert\sin(2\beta)\rvert = \sqrt{1-v^2}$	Proven	Thm. 3.3
$d\tau/dt = m(q_{\mathrm{rel}})$	Defined/Proven	Thm. 3.4
$v = n_\parallel^2 + (1-n_\parallel^2)\cos\eta$	Proven	Thm. 5.2
$\|\Omega\|^2 = 4\pi^2$ (geodesic)	Numerical	Thm. 4.4
Hopf metric decomposition	Numerical (10^{-13})	Eq. (22)
$V_6/V_3 = \pi^2/8$	Proven	Lem. 6.2
$\Omega_0 = \pi^3/4$	Derived	Thm. 6.3
Neutrino $P_{ee} = (1+v)/2$	Proven	Prop. 7.1

12 Conclusion

We have shown that relativistic proper-time dilation is a geometric consequence of dual-observer phase dynamics on the compact manifold $S^3 \cong \mathrm{SU}(2)$. The argument ran from the algebraic obstruction, through the relational construction that supplies the missing structure, to the normalization that ties the kinematics back to the monad geometry. The key results are:

(i) **Lorentz lapse from Hopf geometry.** The function $\sqrt{1-v^2}$ is not a dynamical law but a projection identity: the complementary amplitude of a cosine mode on S^3.

(ii) **Relational observables only.** No physical quantity can be read from a single observer. The relational quaternion $q_{\mathrm{rel}} = q^{(-)}\overline{q^{(+)}}$ is the fundamental observable object, consistent with the coherence identity $C \circ P = I$.

(iii) **Constant internal motion.** The arc-speed $\|\Omega\| = 2\pi$ on S^3 is constant. All kinematic variation in $v(t)$ is projection geometry.

(iv) **Derived normalization.** The constant $\Omega_0 = \pi^3/4$ is forced by the volume ratio $V_6/V_3 = \pi^2/8$ between the six-dimensional generator space and the three-dimensional observable, with no fitting.

(v) α **is coupling, not kinematics.** The fine-structure constant enters through the phase budget of the monad density $\rho(x)$, the coupling between geometric strata, not through the arc-speed or the Hopf projection.

(vi) **Standing wave.** The full structure is a standing wave on S^3: physics is reading the amplitude of a single harmonic on the group manifold.

The framework is not analogous to known physics; it *contains* it. Two-flavor neutrino oscillations are the exact physical realization of Hopf/Bloch precession on S^3, and the survival probability is $(1 + v)/2$. The upstream embedding into $J_3(\mathbb{O})$ (27 dimensions) provides a natural home for the full structure, with the dimension cascade $27 \to 6 \to 3 \to 1$ encoding the progressive projection from algebra to observable.

Two questions were left open here and recorded in the registry: the status of the 2π arc-speed against its tempered value, and the right by which the Hopf observable is called a physical velocity. Both closed downstream, through the unit bridge of Paper 35 and the fold clock of Paper 39 respectively (§10). The resolutions required new structure to be named, not results here to be retracted.

Relativistic spacetime is not fundamental. It is the shadow of compact phase geometry.

References

[1] L. F. Vlegels, *The Monad Rosetta Map: From Void to Structure*, This volume (2025).

[2] L. F. Vlegels, *The Perfect Stable Sphere: Deriving α^{-1} from (B^4, S^3) Geometry*, This volume (2025).

[3] L. F. Vlegels, *The Three-Layer Ontology of Physical Reality*, This volume (2025).

[4] L. F. Vlegels, *The Self-Referential Observation Framework: Particle Masses as Geometric Eigenvalues*, This volume (2025).

[5] L. F. Vlegels, *The Master Operator: Assembling the Geometric Spectral Engine*, This volume (2025).

[6] L. F. Vlegels, *The Unified Field Equation from Geometric First Principles*, This volume (2025).

[7] H. Hopf, "Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche," *Math. Ann.* **104**, 637–665 (1931).

[8] H. K. Urbantke, "The Hopf fibration—seven times in physics," *J. Geom. Phys.* **46**, 125–150 (2003).

[9] J. C. Baez, "The octonions," *Bull. Amer. Math. Soc.* **39**, 145–205 (2002).

[10] P. Jordan, J. von Neumann, and E. Wigner, "On an algebraic generalization of the quantum mechanical formalism," *Ann. Math.* **35**, 29–64 (1934).

[11] J. Maldacena and L. Susskind, "Cool horizons for entangled black holes," *Fortschr. Phys.* **61**, 781–811 (2013).