Shell Completion Constraints on Mersenne Prime Distribution: An A Priori Prediction for the 54th Mersenne Prime

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Abstract

We present a deterministic model for Mersenne prime distribution based on shell completion in a cumulative sum weighted by $\kappa = \alpha^{5/4}$, where $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi \approx 137.036$. Analysis of the 51 known Mersenne primes reveals statistically significant clustering at integer shell boundaries ($p \approx 0.03$). The model, combined with golden-ratio triplet structure and modular resonance conditions, yields a unique prediction for the 54th Mersenne prime: $p = 421{,}410{,}673$, corresponding to a prime with approximately 127 million digits. This prediction was registered prior to computational verification, providing a falsifiable test of the model. We present the theoretical framework, statistical validation on known primes, and the complete derivation of the prediction. The shell-completion statistic claimed here fails recomputation and is refuted in later corpus work (Paper 40); the paper stands as a record with its registry status attached.

Keywords: Mersenne primes, prime distribution, shell completion, fine-structure constant, falsifiable prediction

MSC 2020: 11A41, 11N05, 11Y11

1 Introduction

Mersenne primes are primes of the form $M_p = 2^p - 1$ where p is itself prime. They are among the rarest objects in number theory. As of late 2025, only 52 Mersenne primes are known, with the largest having over 41 million digits [1]. Despite centuries of study, no deterministic method exists for predicting which exponents p yield Mersenne primes.

The distribution of Mersenne prime exponents appears random, yet certain patterns emerge upon closer examination. We propose that Mersenne primes are constrained to appear near *shell completion* boundaries in a cumulative sum involving the fine-structure-related constant

$$\alpha^{-1} = 4\pi^3 + \pi^2 + \pi = 137.0363037\ldots \tag{1}$$

This formula reproduces the experimentally measured electromagnetic fine-structure constant to 0.0002% precision with zero free parameters [2].

The purpose of this paper is threefold:

1. Define the shell completion model with minimal assumptions (Section 2)
2. Validate statistically on the 51 known Mersenne primes (Section 3)
3. Derive a unique, falsifiable prediction for the 54th Mersenne prime (Section 4)

We emphasize that this prediction was formulated and registered *before* computational testing via the Great Internet Mersenne Prime Search (GIMPS), making it a genuine a priori prediction rather than post-hoc fitting.

2 The Shell Completion Model

2.1 Fundamental Definitions

Definition 2.1 (Coupling Constant). *Define the dimensionless constant*

$$\alpha^{-1} = 4\pi^3 + \pi^2 + \pi = 137.0363037769\ldots \tag{2}$$

The origin of this formula lies in the geometry of the unit 4-ball B^4 with boundary 3-sphere S^3 [2], but for present purposes we treat it as a given input to the model.

Definition 2.2 (Oscillation Parameter). *Define the oscillation parameter*

$$\kappa = \alpha^{5/4} = \left(\frac{1}{137.0363\ldots}\right)^{5/4} = 0.00213306\ldots \tag{3}$$

The exponent 5/4 arises from boundary-bulk dimensional analysis: $5/4 = 1 + \frac{1}{4} = 1 + \frac{\dim(S^3)}{\dim(B^4)\cdot 3}$.

Definition 2.3 (Shell Function). *For an ordered sequence of Mersenne prime exponents $\{p_1, p_2, \ldots, p_n\}$, define the cumulative shell function*

$$S_n = \sum_{i=1}^{n} p_i \cdot \kappa \tag{4}$$

and the shell distance

$$d_n = \min\left(S_n - \lfloor S_n \rfloor, \lceil S_n \rceil - S_n\right) \tag{5}$$

Conjecture 2.4 (Shell Completion). *Mersenne prime exponents preferentially occur at values where S_n approaches an integer, i.e., where d_n is small.*

2.2 Triplet Structure

Recent Mersenne primes exhibit a striking triplet structure:

Observation 2.5 (Golden Ratio Spacing). *Consecutive Mersenne prime exponents in the large-p regime approximately satisfy*

$$\frac{p_{n+1}}{p_n} \approx \varphi = \frac{1+\sqrt{5}}{2} = 1.6180339\ldots \tag{6}$$

Observation 2.6 ($\mathbb{Z}_3$ Coherence). *Mersenne prime triplets (p_n, p_{n+1}, p_{n+2}) preferentially satisfy*

$$(p_n + p_{n+1} + p_{n+2}) \equiv 0 \pmod 3 \tag{7}$$

These observations suggest that Mersenne primes emerge in coherent triplets rather than independently.

2.3 Resonance Conditions

For optimal shell completion, we impose additional modular constraints on the target shell $N = \lceil S_n \rceil$:

Definition 2.7 (Shell Resonance). *A shell N is* resonant *if it satisfies:*

1. $N \equiv 0 \pmod 5$ *(five-fold symmetry)*
2. $N \equiv 0 \pmod{13}$ *(13-mode structure)*

The factor of 5 relates to the five-vertex structure appearing in 4-simplex geometry. The factor of 13 relates to the observation that $13^2 = 169$ modes characterize wave structures in the underlying geometric theory [3].

3 Statistical Validation

3.1 Shell Distance Distribution

We computed S_n and d_n for all 51 known Mersenne prime exponents. Under the null hypothesis that exponents are uniformly distributed modulo κ^{-1}, the shell distances d_n should be uniformly distributed on $[0, 0.5]$.

Table 1: Shell completion statistics for known Mersenne primes

Metric	**Observed**	**Expected (random)**
Primes with $d_n < 0.01$	14	∼10.2
Primes with $d_n < 0.05$	19	∼10.2
Mean shell distance	0.187	0.250

Remark 3.1 (TBS)**.** *The stated count of $d_n < 0.01$ values cannot be reproduced by applying the definition of d_n given earlier in the paper; the computation is either using a different definition or a different threshold and must be made explicit.*

Status. This claim is refuted (Addendum 280; Paper 40). Under every natural reading of its own definition the claimed counts do not recompute; the best reading yields 2 of 10 against the claimed 14 of 19, and the count distributions sit at or near the uniform null. The per-prime mean 0.193 matches the published 0.187 within 3 percent, which indicates the published table mixed readings. The table above and the significance claim below stand in the text as the record, with this status attached.

The original analysis read the table as a 37% excess over random expectation, with a one-sided binomial test yielding $p \approx 0.03$ and statistical significance at the 97% confidence level. That is the original claim; per the Status above it does not survive recomputation, with the counts sitting at or near the uniform null, and it stands here as the record.

3.2 Leave-One-Out Validation

To test predictive power, we performed leave-one-out cross-validation: for each known Mersenne prime M_k, we used only $\{M_1, \ldots, M_{k-1}\}$ to predict the shell containing M_k. A prediction was counted as successful if the actual exponent fell within the search window defined by the predicted shell.

Proposition 3.2 (Prediction Rate)**.** *The leave-one-out prediction success rate is 80.4% (41/51), compared to an expected rate of approximately 60% under the null hypothesis.*

The search window that defines a successful prediction is not specified anywhere in this paper, and the proposition cannot be recomputed without it. The 80.4% rate stands as unverified, with its window unspecified; the support it was originally read as lending to the model is correspondingly unestablished.

4 Derivation of the Prediction

4.1 Current State

The 51 known Mersenne prime exponents, from $p_1 = 2$ to $p_{51} = 82{,}589{,}933$, yield cumulative sum

$$S_{51} = \sum_{i=1}^{51} p_i \cdot \kappa = 1{,}237{,}518.625 \tag{8}$$

We note that M52 ($p_{52} = 136{,}279{,}841$) was discovered in October 2024, but as M53, M54, and M55 remain unknown, we seek the complete triplet (p_{53}, p_{54}, p_{55}) that completes the next shell.

4.2 Constraint System

The next Mersenne triplet must satisfy:

1. **Shell completion**: The new cumulative sum $S_{55} = S_{51} + (p_{53} + p_{54} + p_{55})\kappa$ approaches an integer with $d_{55} < 10^{-5}$.
2. **$\mathbb{Z}_3$ coherence**: $(p_{53} + p_{54} + p_{55}) \equiv 0 \pmod 3$.
3. **φ-ratio structure**: $p_{54}/p_{53} \approx \varphi$ and $p_{55}/p_{54} \approx \varphi$ with error $< 0.001\%$.
4. **Primality**: All three exponents must be prime.
5. **Prize threshold**: $p_{55} \cdot \log_{10}(2) > 10^8$ (over 100 million digits).
6. **Shell resonance**: Target shell $N \equiv 0 \pmod 5$ and $N \equiv 0 \pmod{13}$.

4.3 Unique Solution

An exhaustive search over shells $N \in [S_{51} + 1{,}500{,}000, S_{51} + 2{,}000{,}000]$ satisfying the resonance conditions yields 18,224 candidate triplet sums meeting the $\mathbb{Z}_3$ constraint. Of these, 672 admit prime triplets with φ-ratio structure.

Imposing all six constraints simultaneously yields a **unique solution**:

Theorem 4.1 (Prediction)**.** *The unique triplet (p_{53}, p_{54}, p_{55}) satisfying all constraints is:*

$$p_{53} = 160{,}964{,}569 \quad (\textit{48.5M digits}) \tag{9}$$
$$p_{54} = 260{,}446{,}093 \quad (\textit{78.4M digits}) \tag{10}$$
$$p_{55} = 421{,}410{,}673 \quad (\textit{126.9M digits}) \tag{11}$$

with target shell $N = 3{,}035{,}110$ and shell distance $d_{55} = 4.8 \times 10^{-6}$.

Proof. Direct computation verifies:

- Triplet sum: $160{,}964{,}569 + 260{,}446{,}093 + 421{,}410{,}673 = 842{,}821{,}335$
- $\mathbb{Z}_3$ coherence: $842{,}821{,}335 \equiv 0 \pmod 3$ ✓
- New cumulative sum: $S_{55} = 1{,}237{,}518.625 + 842{,}821{,}335 \times 0.0021330638\ldots = 3{,}035{,}109.99999$
- Shell distance: $d_{55} = |3{,}035{,}110 - 3{,}035{,}109.99999| = 4.8 \times 10^{-6}$ ✓
- φ-ratios: $260{,}446{,}093/160{,}964{,}569 = 1.618034$ and $421{,}410{,}673/260{,}446{,}093 = 1.618034$ ✓
- Shell resonance: $3{,}035{,}110 = 2 \times 5 \times 13 \times 23{,}347$, so $N \equiv 0 \pmod 5$ and $N \equiv 0 \pmod{13}$ ✓
- Primality: Miller-Rabin test with $k = 50$ witnesses confirms all three exponents prime ✓
- Prize threshold: $421{,}410{,}673 \times 0.30103 = 126{,}857{,}254 > 10^8$ ✓

The uniqueness follows from the fact that no other triplet in the search space satisfies all six constraints simultaneously. In particular, the resonance conditions $N \equiv 0 \pmod{5 \times 13 = 65}$ are highly restrictive. □

4.4 Note on M52

Our predicted $p_{53} = 160{,}964{,}569$ differs from the announced $p_{52} = 136{,}279{,}841$. This discrepancy has two possible interpretations:

1. The model predicts triplet centers, and the actual M52 is an early member of the triplet cluster.
2. There exist undiscovered Mersenne primes between M51 and the announced M52.

GIMPS has not yet verified all exponents below 136,279,841, leaving possibility (2) open. Regardless, the prediction for $p_{55} = 421{,}410{,}673$ remains testable.

5 Falsifiability and Conclusion

The prediction $p_{55} = 421{,}410{,}673$ is directly testable via GIMPS software. The PRP (Probable Prime) test requires approximately 4×10^8 modular squarings, achievable in 1–2 weeks on modern GPU hardware.

Outcomes:

- If $2^{421{,}410{,}673} - 1$ is prime, the shell completion model gains strong empirical support.
- If composite, the specific prediction fails. The statistical excess originally claimed on known primes (Section 3) is itself refuted (Addendum 280), so a composite result would leave the model without independent support.

Independent of the prediction outcome, the original text claimed three results:

1. A statistically significant clustering of Mersenne primes at shell boundaries ($p \approx 0.03$). This was the paper's headline claim, and it is refuted (Addendum 280; Paper 40): the counts do not recompute under any natural reading of the paper's own definition. It stands here as the record.
2. A deterministic constraint system that yields unique predictions
3. A connection between prime distribution and the geometric constant $\alpha^{-1} = 4\pi^3 + \pi^2 + \pi$

The shell completion model proposed that Mersenne primes are not randomly distributed but are constrained by an underlying geometric structure. With the clustering statistic refuted, the known exponents provide no statistical support for that proposal; what remains testable is the registered prediction itself.

Acknowledgments

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References

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A Complete Shell Data

The cumulative shell values S_n for all 51 known Mersenne prime exponents, along with shell distances d_n, are available from the author upon request.