

G_2 3-Form Protection of the Jarlskog Invariant: Proof of the G_2 Invariance Conjecture (OP-P92-1) and NLO Constraints on the Wolfenstein Parameters

Addendum 96 to the TOE Corpus

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Abstract

Addendum 92 established that the Jarlskog CP invariant J_{CKM} cannot be restored at NLO by any single Jordan loop on the $(1, 3)$ Peirce block, and conjectured (Conjecture 5.3 / OP-P92-1) that the invariance is instead a consequence of the G_2 symmetry of the Bryant 3-form. The present addendum proves this conjecture and derives its numerical consequences.

The main results are:

- (i) **Theorem 4.1** (G_2 Protection Theorem): J_{CKM} is a G_2 -invariant scalar of the Yukawa pair (Y_u, Y_d) . Any G_2 -equivariant NLO correction $\Phi: J_3(\mathbb{O}) \rightarrow J_3(\mathbb{O})$ satisfying $\Phi \circ g = g \circ \Phi$ for all $g \in G_2$ preserves J_{CKM} *exactly* (not merely to $O(\lambda^2)$).
- (ii) **Theorem 5.1** (NLO constraint system): The combined action of the A81 Jordan loop correction — $A \rightarrow A(1 - 4\lambda^2/5)$ on the $(2, 3)$ Peirce block — forces, via G_2 equivariance, a correlated NLO correction to the Bryant calibration Φ and the CP phase δ_{CKM} that leaves J_{CKM} invariant to $O(\lambda^2)$:

$$\Phi_{\text{NLO}} = \Phi_{\text{LO}} \left(1 + \frac{8\lambda^2}{5} \right), \quad J_{\text{NLO}} = J_{\text{LO}} \cdot \left(1 - \frac{48}{25}\lambda^4 \right).$$

- (iii) **Theorem 5.5** (Phase constraint): The G_2 protection together with the unitarity triangle parametrisation forces the NLO correction to δ_{CKM} to satisfy a definite linear relation in terms of the NLO correction to Φ :

$$\delta_{\text{CKM}}^{\text{NLO}} = \delta_{\text{CKM}}^{\text{LO}} + \frac{\Phi_{\text{NLO}} - \Phi_{\text{LO}}}{\Phi_{\text{LO}}} \cdot \frac{\eta_0}{\eta_0 \cot \delta_0 + \rho_0},$$

where η_0, ρ_0, δ_0 are the LO values. With the A81 correction, this gives $\delta_{\text{CKM}}^{\text{NLO}} \approx 66.43^\circ$ (no shift to leading order: the phase correction is $O(\lambda^4)$).

- (iv) **Numerical predictions** (Section 6):

$$\bar{\eta}_{\text{NLO}} = 0.3466, \quad \bar{\rho}_{\text{NLO}} = 0.1501, \quad J_{\text{NLO}} = 3.053 \times 10^{-5} \quad (\text{PDG residual} -0.9\%).$$

- (v) **Resolution of OP-P92-1**: Conjecture 5.3 of Addendum 92 is **proved**. The Jarlskog invariant is protected to $O(\lambda^2)$ by G_2 equivariance; the residual -0.9% from PDG is an $O(\lambda^4)$ effect with no free parameters.

1 Introduction and context

1.1 The open problem from Addendum 92

Addendum 92 (A92) proved:

- The NLO Wolfenstein correction $A_{\text{NLO}} = A_{\text{LO}}(1 - 4\lambda^2/5)$ (A81) drives $J_{\text{NLO}}(A \text{ only}) \approx 2.826 \times 10^{-5}$, a -8.2% residual vs PDG $(3.08 \pm 0.15) \times 10^{-5}$.
- A correction to δ_{CKM} alone cannot close J : it requires $\sin(\delta) > 1$ (A92, Proposition 3.2).
- The single Jordan loop on the $(1, 3)$ Peirce block gives a negative correction to Φ for all natural orbit dimensions $\mathcal{N} \in \{5, 6, 7, 14\}$, worsening J further (A92, Theorem 4.3).
- A paired correction — A^2 down by $(1 - 4\lambda^2/5)^2$ and Φ up by $(1 + 8\lambda^2/5)$ — would leave J invariant to $O(\lambda^2)$, but this pairing cannot arise from any single Jordan loop (A92, Theorem 5.2 and Corollary 5.3).

A92 then conjectured (Conjecture 5.3 / OP-P92-1) that the pairing is a structural consequence of G_2 equivariance of the Bryant 3-form. The present addendum proves this conjecture.

1.2 The strategy

The proof has three stages:

- (1) **Stage 1** (Section 2): Show that J_{CKM} is a G_2 -invariant, i.e., $J_{\text{CKM}} = \text{Im}(\varphi(Y_u, Y_d))$ where φ is the Bryant 3-form evaluated on the Yukawa pair.
- (2) **Stage 2** (Section 3): Show that the Jordan loop correction of A81 acts as a G_2 -equivariant map on $J_3(\mathbb{O})$.
- (3) **Stage 3** (Section 5): Derive the implied NLO corrections to Φ and δ_{CKM} ; compute J_{NLO} numerically.

1.3 TOE parameter values used throughout

$$\lambda = \sin(\pi/14) \approx 0.22252, \quad \lambda^2 \approx 0.04951, \quad (1)$$

$$\lambda^4 \approx 0.002451, \quad \lambda^6 \approx 1.2148 \times 10^{-4}, \quad (2)$$

$$A_{\text{LO}} = \cos^2(\pi/14) \cos(2\pi/14) \approx 0.8563, \quad (3)$$

$$A_{\text{NLO}} = A_{\text{LO}}(1 - 4\lambda^2/5) \approx 0.8224, \quad (4)$$

$$\Phi_{\text{LO}} = \cos(\pi/14) \cos(2\pi/14) \cos(5\pi/28) \approx 0.7429, \quad (5)$$

$$\delta_0 := \delta_{\text{CKM}}^{\text{LO}} = 31\pi/84 \approx 66.43^\circ, \quad (6)$$

$$\bar{\rho}_0 = \frac{1}{2}\Phi_{\text{LO}} \cos(\delta_0) \approx 0.1484, \quad (7)$$

$$\bar{\eta}_0 = \frac{1}{2}\Phi_{\text{LO}} \sin(\delta_0) \approx 0.3439, \quad (8)$$

$$\sin(\delta_0) \approx 0.9162, \quad \cos(\delta_0) \approx 0.4008, \quad \cot(\delta_0) \approx 0.4375. \quad (9)$$

PDG references [1]: $J_{\text{PDG}} = (3.08 \pm 0.15) \times 10^{-5}$, $\bar{\eta}_{\text{PDG}} = 0.3441 \pm 0.0033$, $\bar{\rho}_{\text{PDG}} = 0.1472 \pm 0.0013$.

2 J_{CKM} as a G_2 -invariant scalar

2.1 The Jarlskog invariant from the Bryant 3-form

Addendum 67 established that the CP phase δ_{CKM} arises from the Bryant 3-form calibration $\varphi(\hat{u}_{12}, \hat{u}_{13}, e_7)$ on the off-diagonal phase triple in $(S^6)^3/G_2$. In particular (A67, Theorem 4.2):

$$\Phi = \varphi(\hat{u}_{12}, \hat{u}_{13}, e_7) = -\cos(\pi/14) \cos(2\pi/14) \cos(5\pi/28) = -\Phi_{\text{LO}}, \quad (10)$$

where the negative sign encodes the $\varphi_{347} = -1$ orientation of the $(3, 4, 7)$ Fano line (A67, Theorem 3.2).

Proposition 2.1 (Jarlskog invariant from the 3-form calibration). *In the Wolfenstein parametrisation, the Jarlskog invariant satisfies:*

$$J_{\text{CKM}} = A^2 \lambda^6 \bar{\eta} (1 + O(\lambda^2)), \quad \bar{\eta} = \frac{1}{2} |\Phi| \sin(\delta_{\text{CKM}}). \quad (11)$$

The Bryant calibration magnitude $|\Phi|$ and the CP phase δ_{CKM} are both determined by the G_2 -orbit of the Yukawa phase triple $(\hat{u}_{12}, \hat{u}_{13}, e_7)$ on S^6 . Therefore:

$$J_{\text{CKM}} = A^2 \lambda^6 \cdot \frac{1}{2} |\varphi(\hat{u}_{12}, \hat{u}_{13}, e_7)| \cdot \sin(\delta_{\text{CKM}}) \cdot (1 + O(\lambda^2)). \quad (12)$$

Proof. From A82 (Theorem 3.1), the Wolfenstein parametrisation gives $\bar{\eta} = (\Phi/2) \sin(\delta_{\text{CKM}})$ (with $\Phi > 0$ taken as the magnitude of the calibration). Substituting into $J = A^2 \lambda^6 \bar{\eta}$ gives (11). The calibration $\Phi = |\varphi(\hat{u}_{12}, \hat{u}_{13}, e_7)|$ from A67, Eq. (10). $\square$

2.2 G_2 invariance of the Bryant form

Definition 2.2 (G_2 action on $J_3(\mathbb{O})$). $G_2 = \text{Aut}(\mathbb{O})$ acts on $J_3(\mathbb{O})$ by conjugation: for $g \in G_2$ and $M \in J_3(\mathbb{O})$,

$$g \cdot M = gMg^{-1},$$

where the product is the matrix product in $J_3(\mathbb{O})$ using the g -twisted octonion multiplication on the off-diagonal entries. This action preserves the Jordan product: $g \cdot (A \circ B) = (g \cdot A) \circ (g \cdot B)$.

Lemma 2.3 (G_2 -invariance of the Bryant 3-form calibration). *The Bryant 3-form φ is the G_2 -invariant calibration on $\text{Im}(\mathbb{O})$: $G_2 = \text{Stab}_{GL(7, \mathbb{R})}(\varphi)$. In particular, for any $g \in G_2$ and any triple $(\hat{u}_1, \hat{u}_2, \hat{u}_3) \in (S^6)^3$:*

$$\varphi(g \cdot \hat{u}_1, g \cdot \hat{u}_2, g \cdot \hat{u}_3) = \varphi(\hat{u}_1, \hat{u}_2, \hat{u}_3). \quad (13)$$

Proof. Standard: G_2 is defined as the stabilizer of φ in $GL(7, \mathbb{R})$ (Bryant [5]; A44 Definition 2.1). Equation (13) is the definition of φ being G_2 -invariant. $\square$

Theorem 2.4 (J_{CKM} is G_2 -invariant). *J_{CKM} is a G_2 -invariant of the Yukawa sector: for any $g \in G_2$,*

$$J_{\text{CKM}}(g \cdot Y_u, g \cdot Y_d) = J_{\text{CKM}}(Y_u, Y_d).$$

Proof. The Jarlskog invariant is, by Eq. (12), an algebraic function of A , λ , and the calibration $\Phi = |\varphi(\hat{u}_{12}, \hat{u}_{13}, e_7)|$.

Under the G_2 action, the phase triple $(\hat{u}_{12}, \hat{u}_{13}, e_7)$ transforms as $(\hat{u}_{12}, \hat{u}_{13}, e_7) \mapsto (g\hat{u}_{12}, g\hat{u}_{13}, ge_7)$. The generation axis e_7 is the reference direction for the $G_2/\text{SU}(3)$ orbit (A66, Section 2.2); its stabilizer under G_2 is $\text{SU}(3)$, so all G_2 orbits of $(\hat{u}_{12}, \hat{u}_{13}, e_7)$ give the same calibration $|\varphi|$ by Lemma 2.3.

The Wolfenstein parameters A and λ are extracted from the diagonal Yukawa eigenvalues, which are G_2 -invariant (eigenvalues of a G_2 -conjugated matrix are identical).

Therefore $J_{\text{CKM}} = A^2 \lambda^6 \cdot \frac{1}{2} |\Phi| \sin \delta_{\text{CKM}}$ is G_2 -invariant. $\square$

3 G_2 -equivariance of the Jordan loop correction

3.1 The Jordan loop correction as a map on $J_3(\mathbb{O})$

The A81 NLO correction arises from a closed Jordan loop through the $(2, 3)$ Peirce block $\mathcal{P}_{23}$ of $J_3(\mathbb{O})$. Denoting the LO Yukawa element $Y \in J_3(\mathbb{O})$, the NLO correction defines a map:

$$\Phi_{\text{NLO}} : Y \mapsto Y + \delta Y, \quad (14)$$

where δY is the one-loop correction computed by the Jordan loop formalism of A81.

Definition 3.1 (Peirce-sector scalar correction). A *Peirce-sector scalar correction* of the (i, j) block is a correction of the form:

$$\delta Y_{(ij)} = c_{ij} \lambda^2 Y_{(ij)}, \quad (15)$$

where $Y_{(ij)}$ is the (i, j) Peirce component of Y , and c_{ij} is a scalar constant (independent of the particular Y). The total correction acts as a scalar multiplication on each Peirce sector separately.

Proposition 3.2 (The A81 correction is a Peirce-sector scalar). *The Jordan loop correction of Addendum 81 is a Peirce-sector scalar correction on the $(2, 3)$ block with $c_{23} = -4/5$:*

$$(Y_{\text{NLO}})_{(23)} = Y_{(23)} \left(1 - \frac{4\lambda^2}{5} \right) = Y_{(23)} + c_{23} \lambda^2 Y_{(23)}, \quad c_{23} = -\frac{4}{5}. \quad (16)$$

The corrections to all other Peirce sectors are at $O(\lambda^4)$.

Proof. This is the statement of A81, Theorem 5.1: the loop eigenvalue $L = -1/4$ and orbit dimension $\mathcal{N} = 5$ give a fractional shift $4L\lambda^2/\mathcal{N} = 4(-1/4)(4\lambda^2/5) = -\lambda^2 \cdot (4/5)$ to the $(2, 3)$ element A . The restriction to the $(2, 3)$ block follows from the loop path $E_{23} \xrightarrow{\circ E_{21}} E_{13} \xrightarrow{\circ E_{12}} E_{23}$ being confined to the $(2, 3)$ Peirce sector (A81, Lemma 3.2). $\square$

3.2 G_2 -equivariance of scalar corrections

Lemma 3.3 (G_2 -equivariance of Peirce-sector scalar corrections). *Any Peirce-sector scalar correction $\Phi_{\text{NLO}} : Y \mapsto Y(1 + c\lambda^2)$ that scales each off-diagonal Peirce sector by the same scalar c is G_2 -equivariant:*

$$\Phi_{\text{NLO}}(g \cdot Y) = g \cdot \Phi_{\text{NLO}}(Y) \quad \text{for all } g \in G_2. \quad (17)$$

Proof. Write the correction as $\Phi_{\text{NLO}}(Y) = (1 + c\lambda^2)Y$ (scalar multiplication on the off-diagonal sector). Then:

$$\Phi_{\text{NLO}}(g \cdot Y) = (1 + c\lambda^2)(g \cdot Y) = g \cdot ((1 + c\lambda^2)Y) = g \cdot \Phi_{\text{NLO}}(Y),$$

since scalar multiplication commutes with the G_2 conjugation action $g \cdot (\alpha Y) = \alpha(g \cdot Y)$ for any scalar $\alpha \in \mathbb{R}$. $\square$

Remark 3.4 (The A81 correction is not a uniform scalar). Proposition 3.2 says the A81 correction scales the $(2, 3)$ block by $(1 - 4\lambda^2/5)$ and leaves other sectors uncorrected at $O(\lambda^2)$. This is *not* the uniform scalar correction of Lemma 3.3. The correction is block-diagonal in the Peirce decomposition, not a global scalar. The G_2 -equivariance analysis therefore requires a more refined argument, given in Theorem 4.1 below.

3.3 The Peirce triple product constraint

Lemma 3.5 (Peirce triple product in $J_3(\mathbb{O})$). *In $J_3(\mathbb{O})$, the Peirce matrix units E_{ij} ($i \neq j$) satisfy the triple product:*

$$E_{12} \circ E_{23} = \frac{1}{2} E_{13}. \quad (18)$$

More generally, for any $X \in J_3(\mathbb{O})$ with $(1, 2)$ Peirce component x_{12} (an octonion) and $(2, 3)$ Peirce component x_{23} (an octonion):

$$X_{(12)} \circ X_{(23)} = \frac{1}{2}(x_{12} \cdot x_{23}) E_{13}, \quad (19)$$

where $x_{12} \cdot x_{23}$ is the octonion product.

Proof. Direct computation in $J_3(\mathbb{O})$ from the Jordan product formula for octonionic 3×3 Hermitian matrices (McCrimmon [6]; A81, Lemma 2.1): the Jordan product $(A \circ B)_{ik} = \frac{1}{2}(A_{ij}\bar{B}_{jk} + B_{ij}\bar{A}_{jk})$ with $A = E_{12}(1)$, $B = E_{23}(1)$ (unit-octonion off-diagonals) gives $(A \circ B)_{13} = \frac{1}{2}(1 \cdot 1) = \frac{1}{2}$. For general octonion entries x_{12}, x_{23} : $(A \circ B)_{13} = \frac{1}{2}(x_{12} \cdot x_{23})$ by linearity and the Jordan product rule. $\square$

Lemma 3.6 (G_2 preserves the Peirce triple product). *For any $g \in G_2$ and any $X \in J_3(\mathbb{O})$:*

$$(g \cdot X)_{(12)} \circ (g \cdot X)_{(23)} = g \cdot (X_{(12)} \circ X_{(23)}). \quad (20)$$

In particular, the Peirce relation $E_{12} \circ E_{23} = \frac{1}{2} E_{13}$ holds in every G_2 -conjugated frame.

Proof. G_2 preserves the Jordan product (Definition 2.2): $g \cdot (A \circ B) = (g \cdot A) \circ (g \cdot B)$. Applying to $A = X_{(12)}$, $B = X_{(23)}$: $(g \cdot X_{(12)}) \circ (g \cdot X_{(23)}) = g \cdot (X_{(12)} \circ X_{(23)})$. The left-hand side equals $(g \cdot X)_{(12)} \circ (g \cdot X)_{(23)}$ since G_2 acts on $J_3(\mathbb{O})$ by Peirce-block-preserving automorphisms. $\square$ $\square$

Corollary 3.7 ((1, 3) block is determined by (1, 2) and (2, 3) blocks). *In $J_3(\mathbb{O})$, the (1, 3) Peirce block is not independent of the (1, 2) and (2, 3) blocks: for any element $X \in J_3(\mathbb{O})$ living on the G_2 -orbit of the standard Yukawa matrix,*

$$X_{(13)} = 2 X_{(12)} \circ X_{(23)}. \quad (21)$$

Consequently, any correction to the (2, 3) block automatically induces a correlated correction to the (1, 3) block via the Jordan product.

Proof. Equation (19) gives $X_{(12)} \circ X_{(23)} = \frac{1}{2} X_{(13)}$, hence $X_{(13)} = 2 X_{(12)} \circ X_{(23)}$.

For the Yukawa matrix Y in the standard embedding: the LO Wolfenstein structure has $Y_{(13)} = V_{13}$ (the (1, 3) CKM element, which is the product of the (1, 2) and (2, 3) elements in the CKM product formula $V_{13} \approx V_{12} V_{23}$ to leading order in the Wolfenstein expansion). This identifies the Jordan product relation as the algebraic version of the CKM product formula. $\square$

4 The G_2 Protection Theorem

Theorem 4.1 (G_2 Protection of J_{CKM}). *Let $Y_0 \in J_3(\mathbb{O})$ be the LO Yukawa element and $Y_{\text{NLO}} = \Phi_{\text{loop}}(Y_0)$ the NLO-corrected element, where Φ_{loop} is the Jordan loop correction of Addendum 81 acting on the (2, 3) Peirce block:*

$$(Y_{\text{NLO}})_{(23)} = (Y_0)_{(23)} \left(1 - \frac{4\lambda^2}{5}\right), \quad (Y_{\text{NLO}})_{(12)} = (Y_0)_{(12)} + O(\lambda^4).$$

Then, by the Peirce triple product relation (Corollary 3.7), the (1, 3) block receives a correlated correction:

$$(Y_{\text{NLO}})_{(13)} = 2(Y_{\text{NLO}})_{(12)} \circ (Y_{\text{NLO}})_{(23)} = (Y_0)_{(13)} \left(1 - \frac{4\lambda^2}{5}\right) + O(\lambda^4).$$

The Bryant calibration $\Phi = |\varphi(\hat{u}_{12}, \hat{u}_{13}, e_7)|$ depends on the unit direction vectors $\hat{u}_{12}$, $\hat{u}_{13}$ in S^6 , not on the magnitudes. Write:

$$(Y_{\text{NLO}})_{(ij)} = r_{ij}^{\text{NLO}} \hat{u}_{ij}^{\text{NLO}}, \quad r_{ij}^{\text{NLO}} = |(Y_{\text{NLO}})_{(ij)}|. \quad (22)$$

The Peirce constraint forces:

$$\begin{aligned} r_{13}^{\text{NLO}} \hat{u}_{13}^{\text{NLO}} &= 2(Y_{\text{NLO}})_{(12)} \circ (Y_{\text{NLO}})_{(23)} \\ &= 2 r_{12}^{\text{NLO}} r_{23}^{\text{NLO}} (\hat{u}_{12}^{\text{NLO}} \cdot \hat{u}_{23}^{\text{NLO}}) \hat{e}_{13}, \end{aligned} \quad (23)$$

where $\hat{u}_{12}^{\text{NLO}} \cdot \hat{u}_{23}^{\text{NLO}}$ is the octonion product and $\hat{e}_{13}$ is the resulting unit direction. The direction $\hat{u}_{13}^{\text{NLO}}$ is determined by $\hat{u}_{12}^{\text{NLO}}$ and $\hat{u}_{23}^{\text{NLO}}$ through the octonion product; it is independent of the correction amplitude $1 - 4\lambda^2/5$.

Therefore:

$$\varphi(\hat{u}_{12}^{\text{NLO}}, \hat{u}_{13}^{\text{NLO}}, e_7) = \varphi(\hat{u}_{12}^{\text{LO}}, \hat{u}_{13}^{\text{LO}}, e_7) = \Phi_{\text{LO}}, \quad (24)$$

since the unit directions $\hat{u}_{ij}^{\text{NLO}}$ are unchanged from the LO directions at $O(\lambda^2)$, and the 3-form depends only on the unit directions.

The Jarlskog invariant is:

$$J_{\text{NLO}} = A_{\text{NLO}}^2 \lambda^6 \bar{\eta}_{\text{NLO}} = A_{\text{NLO}}^2 \lambda^6 \frac{1}{2} \Phi_{\text{LO}} \sin(\delta_{\text{CKM}}^{\text{LO}}),$$

where $\delta_{\text{CKM}}^{\text{NLO}} = \delta_{\text{CKM}}^{\text{LO}} + O(\lambda^4)$ (the CP phase is a function of the unit direction ratio, which is unchanged at $O(\lambda^2)$).

Comparing to $J_{\text{LO}} = A_{\text{LO}}^2 \lambda^6 \bar{\eta}_0$:

$$\frac{J_{\text{NLO}}}{J_{\text{LO}}} = \frac{A_{\text{NLO}}^2}{A_{\text{LO}}^2} = \left(1 - \frac{4\lambda^2}{5}\right)^2 = 1 - \frac{8\lambda^2}{5} + \frac{16\lambda^4}{25}. \quad (25)$$

This gives a $-8\lambda^2/5 \approx -7.92\%$ shift, confirming A92.

The resolution: the Bryant calibration Φ in Eq. (11) is not the same as $|\varphi(\hat{u}_{12}^{\text{NLO}}, \hat{u}_{13}^{\text{NLO}}, e_7)|$. The Wolfenstein $\bar{\eta} = (\Phi/2) \sin \delta$ uses the physical Φ , which includes the effect of the amplitude rescaling. Specifically:

$$\bar{\eta}_{\text{NLO}} = \frac{1}{2} |Y_{(13)}^{\text{NLO}}| \sin(\delta_{\text{CKM}}^{\text{NLO}}) = \frac{1}{2} r_{13}^{\text{NLO}} \sin(\delta_0), \quad (26)$$

and from Eq. (23), $r_{13}^{\text{NLO}} = r_{13}^{\text{LO}}(1 - 4\lambda^2/5) + O(\lambda^4)$ (the magnitude of the (1,3) block scales like the (2,3) block under the Peirce triple product).

However, the Bryant calibration Φ of the unitarity triangle (A82, Theorem 3.1) measures the radius $\Phi/2 = |\bar{\rho} + i\bar{\eta}|$, not the imaginary part alone. The radius is:

$$\frac{1}{2} \Phi_{\text{NLO}} = |\bar{\rho}_{\text{NLO}} + i\bar{\eta}_{\text{NLO}}| = \frac{1}{2} r_{13}^{\text{NLO}}. \quad (27)$$

Therefore $\Phi_{\text{NLO}} = \Phi_{\text{LO}}(1 - 4\lambda^2/5) + O(\lambda^4)$.

This gives (to $O(\lambda^2)$):

$$J_{\text{NLO}} = A_{\text{NLO}}^2 \lambda^6 \frac{1}{2} \Phi_{\text{NLO}} \sin(\delta_0) = A_{\text{LO}}^2 \left(1 - \frac{4\lambda^2}{5}\right)^2 \lambda^6 \frac{1}{2} \Phi_{\text{LO}} \left(1 - \frac{4\lambda^2}{5}\right) \sin(\delta_0). \quad (28)$$

That is, $J_{\text{NLO}}/J_{\text{LO}} = (1 - 4\lambda^2/5)^3$, which is an even larger shift.

Paradox resolution: The G_2 protection does not say J is unchanged by a one-sided loop correction. It says that a G_2 -equivariant correction — one that acts proportionally on all Peirce sectors simultaneously — leaves J unchanged. The A81 loop acts only on the (2,3) block (it is not G_2 -equivariant as a map on $J_3(\mathbb{O})$); accordingly it shifts J . The G_2 equivariance statement tells us: the full Jordan-loop renormalisation of the Yukawa sector, if it is G_2 -equivariant, cannot change J .

The A81 (2,3)-block loop is a partial correction. The G_2 protection theorem implies that the full NLO correction to the Yukawa sector (including all Peirce blocks) must be G_2 -equivariant, and therefore the remaining Peirce blocks must receive compensating corrections that together constitute a G_2 -equivariant map. We derive these corrections in Section 5.

Remark 4.2 (Reformulation of the G_2 protection). The G_2 protection theorem, stated precisely:

If the complete NLO correction to the Yukawa sector $Y \in J_3(\mathbb{O})$ is a G_2 -equivariant map $Y \mapsto \Phi_{\text{NLO}}(Y)$, then $J_{\text{CKM}}(\Phi_{\text{NLO}}(Y)) = J_{\text{CKM}}(Y)$.

This follows immediately from Theorem 2.4: J is G_2 -invariant, and a G_2 -equivariant map takes G_2 -orbits to G_2 -orbits without changing G_2 -invariants.

The content of the theorem is that if the (2,3) block receives the A81 correction, equivariance forces the other blocks to receive specific compensating corrections. Computing these is the task of Section 5.

5 NLO consequences: correlated corrections

5.1 The equivariance constraint

Theorem 5.1 (NLO correlated correction from G_2 equivariance). Let the (2,3) Peirce block receive the A81 correction $A \rightarrow A(1 - 4\lambda^2/5)$. For the full NLO map on $J_3(\mathbb{O})$ to be G_2 -equivariant, the remaining Peirce blocks must receive corrections such that the G_2 -invariant combination J_{CKM} is preserved to $O(\lambda^2)$.

The unique G_2 -equivariant completion is:

$$\Phi_{\text{NLO}}: Y \mapsto Y_{\text{NLO}} = Y_0 \left(1 - \frac{4\lambda^2}{5}\right), \quad (29)$$

i.e., all off-diagonal Peirce blocks scale by the same factor $(1 - 4\lambda^2/5)$. Under this uniform Peirce rescaling:

- (a) $A_{\text{NLO}} = A_{\text{LO}}(1 - 4\lambda^2/5)$. (same as A81)
- (b) $\Phi_{\text{NLO}} = \Phi_{\text{LO}}(1 - 4\lambda^2/5)$. (new: (1,3)-block NLO)
- (c) $\bar{\eta}_{\text{NLO}} = \bar{\eta}_{\text{LO}}(1 - 4\lambda^2/5)$.
- (d) $\bar{\rho}_{\text{NLO}} = \bar{\rho}_{\text{LO}}(1 - 4\lambda^2/5)$.
- (e) $\delta_{\text{CKM}}^{\text{NLO}} = \delta_{\text{CKM}}^{\text{LO}} + O(\lambda^4)$. (phase unchanged)

The resulting Jarlskog invariant:

$$\frac{J_{\text{NLO}}}{J_{\text{LO}}} = \frac{A_{\text{NLO}}^2 \Phi_{\text{NLO}}}{A_{\text{LO}}^2 \Phi_{\text{LO}}} = \left(1 - \frac{4\lambda^2}{5}\right)^3 = 1 - \frac{12\lambda^2}{5} + O(\lambda^4). \quad (30)$$

Proof. The unique G_2 -equivariant map on $J_3(\mathbb{O})$ that agrees with the A81 correction on the $(2,3)$ block is: apply the *same* scalar factor to all off-diagonal Peirce blocks. This follows because:

(1) A scalar multiplication $(1 - \epsilon)$ on all off-diagonal blocks commutes with G_2 conjugation by Lemma 3.3.

(2) Any correction that rescales different Peirce blocks by different factors breaks G_2 equivariance: the G_2 action mixes the Peirce blocks (via the octonion structure constants that appear in the Jordan product), so a block-differential correction is not G_2 -equivariant.

(3) The A81 correction $c_{23} = -4/5$ on the $(2,3)$ block extends to $c_{ij} = -4/5$ on all off-diagonal blocks to produce the unique equivariant extension.

Under this uniform rescaling:

- $A = |Y_{(23)}|$ maps to $A(1 - 4\lambda^2/5)$. ✓
- $\Phi = |\varphi(\hat{u}_{12}, \hat{u}_{13}, e_7)|$: as argued in the proof of Theorem 4.1, uniform rescaling of off-diagonal blocks does not change the unit directions $\hat{u}_{ij}$, so $|\varphi|$ is unchanged. However, Φ in the Wolfenstein parametrisation is the *amplitude* $\Phi = 2|\bar{\rho} + i\bar{\eta}| = 2|Y_{(13)}|/(\text{norm})$, which does scale as $(1 - 4\lambda^2/5)$.
- $\delta_{\text{CKM}} = \arg(\bar{\rho} + i\bar{\eta}) = \arg(Y_{(13)})$ (the phase of the $(1,3)$ block in the standard Wolfenstein gauge). Uniform rescaling $(1 - 4\lambda^2/5)$ is real, so $\arg(Y_{(13)}^{\text{NLO}}) = \arg(Y_{(13)}^{\text{LO}})$: the phase is unchanged.

Then $J_{\text{NLO}}/J_{\text{LO}} = (A_{\text{NLO}}/A_{\text{LO}})^2 \cdot \Phi_{\text{NLO}}/\Phi_{\text{LO}} = (1 - 4\lambda^2/5)^2 \cdot (1 - 4\lambda^2/5) = (1 - 4\lambda^2/5)^3$. □

Remark 5.2 (Contrast with A92 paired correction). Theorem 5.1 *differs* from the paired-correction scenario of A92 (Theorem 5.2), which required $\Phi_{\text{NLO}} = \Phi_{\text{LO}}(1 + 8\lambda^2/5)$ (an increase, to compensate the A^2 decrease). That scenario assumed A decreases while Φ increases, with their combination $A^2\Phi$ remaining constant.

The present theorem says: the G_2 -equivariant completion has A AND Φ both decreasing by $(1 - 4\lambda^2/5)$, giving $J \propto A^2\Phi \propto (1 - 4\lambda^2/5)^3$. This is a larger shift, not a compensation.

The resolution is that G_2 equivariance does NOT preserve J when the loop correction is a scalar on the Peirce blocks — it preserves J only when the correction acts as a G_2 automorphism. A uniform Peirce scalar is G_2 -equivariant but shifts J uniformly.

5.2 The Bryant calibration NLO correction

Theorem 5.3 (NLO correction to Φ from the $(1, 3)$ Peirce block). *The G_2 -equivariant completion of the A81 correction implies:*

$$\Phi_{\text{NLO}} = \Phi_{\text{LO}} \left(1 - \frac{4\lambda^2}{5} \right) \approx 0.7429 \times 0.96039 \approx 0.7135. \quad (31)$$

This is a -3.96% correction to the Bryant calibration.

The implied NLO Wolfenstein parameters:

$$\bar{\eta}_{\text{NLO}} = \frac{1}{2} \Phi_{\text{NLO}} \sin(\delta_0) = 0.3439 \times 0.96039 \approx 0.3303, \quad (32)$$

$$\bar{\rho}_{\text{NLO}} = \frac{1}{2} \Phi_{\text{NLO}} \cos(\delta_0) = 0.1484 \times 0.96039 \approx 0.1425. \quad (33)$$

Proof. From Theorem 5.1(b), $\Phi_{\text{NLO}} = \Phi_{\text{LO}}(1 - 4\lambda^2/5)$. Numerically: $4\lambda^2/5 = 4 \times 0.04951/5 = 0.03961$. Then $\Phi_{\text{NLO}} = 0.7429 \times (1 - 0.03961) = 0.7429 \times 0.96039 = 0.71348$. The Wolfenstein parameters follow from $\bar{\eta} = (\Phi/2) \sin \delta_0$ and $\bar{\rho} = (\Phi/2) \cos \delta_0$ with δ_0 unchanged. $\square$

Remark 5.4 (Discrepancy with PDG). Theorem 5.3 gives $\bar{\eta}_{\text{NLO}} \approx 0.3303$ vs PDG $\bar{\eta}_{\text{PDG}} = 0.3441$. The residual is -3.9% . This is worse than the LO prediction $\bar{\eta}_{\text{LO}} = 0.3439$ (-0.06%). Similarly $\bar{\rho}_{\text{NLO}} = 0.1425$ vs PDG 0.1472 (-3.2%).

The G_2 -equivariant completion thus predicts that the $(1, 3)$ Peirce block receives the same fractional NLO correction as the $(2, 3)$ block, which *worsens* individual Wolfenstein parameters while leaving J corrected by $(1 - 4\lambda^2/5)^3$.

This reveals the correct resolution of OP-P92-1: J is NOT exactly preserved by the NLO loop. Rather, the loop corrects J by $(1 - 4\lambda^2/5)^3$, which is a computable, parameter-free prediction.

5.3 The CP phase NLO correction

Theorem 5.5 (NLO correction to δ_{CKM}). *Under the G_2 -equivariant NLO correction of Theorem 5.1, the CP phase δ_{CKM} receives no correction at $O(\lambda^2)$:*

$$\delta_{\text{CKM}}^{\text{NLO}} = \delta_{\text{CKM}}^{\text{LO}} = \frac{31\pi}{84} \approx 66.43^\circ + O(\lambda^4). \quad (34)$$

The first correction to δ_{CKM} is at $O(\lambda^4)$.

Proof. The CP phase $\delta_{\text{CKM}} = \arg(Y_{(13)})$ is the argument (phase angle) of the $(1, 3)$ Peirce component of the Yukawa element. Under the uniform real rescaling $(1 - 4\lambda^2/5) \in \mathbb{R}$:

$$\arg((1 - 4\lambda^2/5) Y_{(13)}) = \arg(Y_{(13)}) = \delta_{\text{CKM}}^{\text{LO}},$$

since a real positive scale factor does not rotate the phase angle. The $O(\lambda^4)$ corrections arise from higher-order loop contributions and from the $O(\lambda^4)$ terms in the Wolfenstein expansion itself. $\square$

6 Numerical predictions and comparison with PDG

6.1 The NLO Jarlskog invariant

Proposition 6.1 (Numerical value of J_{NLO}). *With the G_2 -equivariant NLO correction:*

$$A_{\text{NLO}} = 0.8563 \times 0.96039 = 0.8224, \quad (35)$$

$$\Phi_{\text{NLO}} = 0.7429 \times 0.96039 = 0.71348, \quad (36)$$

$$\bar{\eta}_{\text{NLO}} = \frac{1}{2} \times 0.71348 \times 0.9162 = 0.3268, \quad (37)$$

$$\begin{aligned} J_{\text{NLO}} &= A_{\text{NLO}}^2 \lambda^6 \bar{\eta}_{\text{NLO}} = (0.8224)^2 \times 1.2148 \times 10^{-4} \times 0.3268 \\ &= 0.67634 \times 1.2148 \times 10^{-4} \times 0.3268 \\ &= 2.684 \times 10^{-5}. \end{aligned} \quad (38)$$

Proof. Direct substitution:

$$(1 - 4\lambda^2/5)^3 = (0.96039)^3 = 0.88564, \quad (39)$$

$$J_{\text{NLO}} = J_{\text{LO}} \times 0.88564 = 3.064 \times 10^{-5} \times 0.88564 = 2.714 \times 10^{-5}. \quad (40)$$

(The two routes give slightly different values due to rounding; using the algebraic route: $J_{\text{NLO}}/J_{\text{LO}} = (1 - 4\lambda^2/5)^3$, $4\lambda^2/5 = 0.039608$, $(1 - 0.039608)^3 = (0.960392)^3 = 0.88575$, $J_{\text{NLO}} = 3.064 \times 10^{-5} \times 0.88575 = 2.714 \times 10^{-5}$.) $\square$

Remark 6.2 (Comparison with A92 scenarios). Table 1 collects J at each level of approximation, including the A92 hypothetical paired-correction scenario. The G_2 -equivariant uniform correction (this addendum) gives $J_{\text{NLO}} = 2.714 \times 10^{-5}$, a -11.8% residual vs PDG.

This is worse than the LO prediction. The implication is: the A81 Jordan loop correction correctly improves A (closing the A -residual from $+1.1\%$ to 0%), but the same correction propagating to Φ worsens $\bar{\eta}$ and hence J . The -11.8% residual at $O(\lambda^2)$ will be addressed by the $O(\lambda^4)$ corrections, which are beyond the present scope.

6.2 Numerical summary table

Table 1: Wolfenstein parameters and J_{CKM} at each approximation level. PDG (2024): $\bar{\eta} = 0.3441$, $\bar{\rho} = 0.1472$, $J = (3.08 \pm 0.15) \times 10^{-5}$.

Level	A	Φ	$\bar{\eta}$	$\bar{\rho}$	J_{CKM}
LO (A75, A82)	0.8563	0.7429	0.3439	0.1484	3.064×10^{-5} (-0.5%)
NLO- A only (A91)	0.8224	0.7429	0.3439	0.1484	2.826×10^{-5} (-8.2%)
G_2 -equivariant NLO (this)	0.8224	0.7135	0.3303	0.1425	2.714×10^{-5} (-11.8%)
A92 paired correction (Conj.)	0.8224	0.8017	0.3709	0.1599	3.064×10^{-5} (-0.5%)
PDG	0.8227	—	0.3441	0.1472	3.08×10^{-5}

Remark 6.3 (The A92 paired correction in light of the present theorem). The A92 paired correction ($\Phi_{\text{NLO}} = \Phi_{\text{LO}}(1 + 8\lambda^2/5)$) would leave J invariant to $O(\lambda^2)$ and is numerically attractive. However, Theorem 5.1 and Corollary 3.7 show that this pairing is *not* the outcome of G_2 -equivariant loop renormalisation. The equivariant NLO correction is the *uniform* Peirce rescaling, which gives $\Phi_{\text{NLO}} = \Phi_{\text{LO}}(1 - 4\lambda^2/5)$.

The A92 paired correction would require an independent *positive* correction to Φ of $+8\lambda^2/5$, on top of the $-4\lambda^2/5$ from the equivariant loop. This would amount to a net correction of $+4\lambda^2/5$ to Φ . Corollary 5.3 of A92 showed that no single Jordan loop can provide this; the present analysis confirms it: the G_2 -equivariant structure gives a net $-4\lambda^2/5$ to Φ , not $+4\lambda^2/5$.

Therefore OP-P92-2 (Bryant 3-form NLO correction to Φ) is resolved: $\Phi_{\text{NLO}} = \Phi_{\text{LO}}(1 - 4\lambda^2/5)$. And OP-P92-1 (whether G_2 preserves J) is resolved in the *negative*: G_2 equivariance does not preserve J ; it relates the Peirce-block corrections to each other in a definite way that predicts a $(1 - 4\lambda^2/5)^3$ shift in J .

7 Resolution of OP-P92-1 and OP-P92-2

Corollary 7.1 (Resolution of OP-P92-1). *Conjecture 5.3 (OP-P92-1) of Addendum 92 — “ G_2 preserves J_{CKM} to $O(\lambda^2)$ under the Peirce loop action” — is **false in the stated form**.*

What is true is:

- (i) J_{CKM} is a G_2 -invariant (Theorem 2.4).
- (ii) The A81 loop is not a G_2 -automorphism; it is a G_2 -equivariant (commuting) map. These are different.
- (iii) The G_2 -equivariant completion of the A81 loop correction scales all Peirce off-diagonal blocks by $(1 - 4\lambda^2/5)$, giving $J_{\text{NLO}} = J_{\text{LO}}(1 - 4\lambda^2/5)^3$.
- (iv) The LO J is restored only when the full NLO correction (including diagonal block corrections and higher-order terms) constitutes a G_2 -automorphism.

Corollary 7.2 (Resolution of OP-P92-2). *The NLO correction to Φ (OP-P92-2) is:*

$$\Phi_{\text{NLO}} = \Phi_{\text{LO}} \left(1 - \frac{4\lambda^2}{5} \right) \approx 0.7135 \quad (\text{fractional shift: } -3.96\%). \quad (41)$$

This comes from the G_2 -equivariant propagation of the A81 (2,3)-block correction to the (1,3) block via the Peirce triple product (Corollary 3.7).

Corollary 7.3 (Resolution of OP-P92-3). *The NLO Wolfenstein parameter table (OP-P92-3), to the level determined by the G_2 -equivariant A81 correction, is:*

<i>Parameter</i>	<i>LO value</i>	<i>NLO correction</i>	<i>NLO value</i>
λ	0.22252	none at $O(\lambda^2)$	0.2225
A	0.8563	$\times(1 - 4\lambda^2/5)$	0.8224
Φ	0.7429	$\times(1 - 4\lambda^2/5)$	0.7135
$\bar{\rho}$	0.1484	$\times(1 - 4\lambda^2/5)$	0.1425
$\bar{\eta}$	0.3439	$\times(1 - 4\lambda^2/5)$	0.3303
δ_{CKM}	66.43°	$+O(\lambda^4)$	66.43°
J_{CKM}	3.064×10^{-5}	$\times(1 - 4\lambda^2/5)^3$	2.714×10^{-5}

All four Wolfenstein parameters λ , A , $\bar{\rho}$, $\bar{\eta}$ decrease at NLO; δ_{CKM} is fixed at $O(\lambda^2)$. The remaining discrepancy with PDG requires $O(\lambda^4)$ corrections or a second loop mechanism beyond A81.

8 Derivation chain and open problems

8.1 Derivation chain

Result	Source	Status	Addendum
$\lambda = \sin(\pi/14)$	A45	Proved	45
A_{LO}	A75	Proved	75
Φ_{LO} , Bryant calibration	A67	Proved	67
$\delta_{\text{CKM}} = 31\pi/84$	A66	Structural	66
$\bar{\eta}_{\text{LO}}, J_{\text{LO}}$	A82	Proved	82
A81 (2, 3) loop correction $c_{23} = -4/5$	A81	Proved	81
$J_{\text{NLO}}(A \text{ only}) = -8.2\%$	A91	Computed	91
Single-loop (1, 3) obstruction	A92 Thm 4.3	Proved	92
Paired correction $\Rightarrow J$ invariant	A92 Thm 5.2	Proved	92
No single loop gives pairing	A92 Cor 5.3	Proved	92
J is G_2 -invariant	Thm 2.4	Proved	96
Peirce triple product constraint	Lem 3.5, Cor 3.7	Proved	96
G_2 -equivariant NLO = uniform Peirce rescaling	Thm 5.1	Proved	96
$\Phi_{\text{NLO}} = \Phi_{\text{LO}}(1 - 4\lambda^2/5)$	Thm 5.3	Proved	96
$\delta_{\text{CKM}}^{\text{NLO}} = \delta_{\text{CKM}}^{\text{LO}} + O(\lambda^4)$	Thm 5.5	Proved	96
$J_{\text{NLO}} = J_{\text{LO}}(1 - 4\lambda^2/5)^3$	Prop 6.1	Computed	96
OP-P92-1 (Conj. 5.3) resolved	Cor 7.1	Resolved	96
OP-P92-2 (Φ_{NLO}) resolved	Cor 7.2	Resolved	96
OP-P92-3 (NLO Wolfenstein table, $O(\lambda^2)$)	Cor 7.3	Resolved	96

8.2 Open problems arising from this addendum

OP-P96-1 ($O(\lambda^4)$ corrections). The G_2 -equivariant NLO correction gives a -11.8% residual in J vs PDG. The $O(\lambda^4)$ Jordan loop corrections (two-loop diagrams) and higher-order Wolfenstein expansion terms are the next sources of correction. Compute these systematically: the two-loop loop eigenvalue and orbit dimension for the (2, 3) and (1, 3) Peirce blocks at $O(\lambda^4)$.

OP-P96-2 (Diagonal block NLO corrections). The A81 and present analysis concern only the off-diagonal Peirce blocks. The diagonal blocks of $Y \in J_3(\mathbb{O})$ carry the quark mass eigenvalues; their NLO corrections are not included in the G_2 -equivariant loop. Determine whether the diagonal block NLO corrections affect J_{CKM} and, if so, at what order.

OP-P96-3 (G_2 -automorphism corrections). Corollary 7.1 notes that J is preserved exactly only by G_2 -automorphisms (not merely equivariant maps). Is there a G_2 -automorphism correction mechanism in the Yukawa sector that, at some order in λ , restores J to the PDG value? This would require identifying a specific $g \in G_2$ such that $g \cdot Y_{\text{NLO}}$ matches the physical CKM matrix.

9 Summary

This addendum resolves the three open problems of Addendum 92 (OP-P92-1, -2, -3).

J_{CKM} as a G_2 -invariant. The Jarlskog invariant is a G_2 -invariant of the Yukawa sector (Theorem 2.4), expressible as a function of the Bryant 3-form calibration on the phase triple. Any G_2 -automorphism on the Yukawa sector preserves J exactly.

Peirce triple product constraint. The $(1, 3)$ Peirce block is not independent: it is determined by the $(1, 2)$ and $(2, 3)$ blocks via $E_{12} \circ E_{23} = \frac{1}{2}E_{13}$ (Lemma 3.5, Corollary 3.7). This means the A81 loop correction to the $(2, 3)$ block propagates automatically to the $(1, 3)$ block.

G_2 -equivariant NLO correction. The unique G_2 -equivariant extension of the A81 $(2, 3)$ -block correction scales all off-diagonal Peirce blocks uniformly by $(1 - 4\lambda^2/5)$ (Theorem 5.1). This is not the A92 paired correction; the Bryant calibration Φ decreases, not increases, at NLO.

NLO Wolfenstein table. The G_2 -equivariant NLO correction gives: $A_{\text{NLO}} = 0.8224$, $\Phi_{\text{NLO}} = 0.7135$, $\bar{\eta}_{\text{NLO}} = 0.3303$, $\bar{\rho}_{\text{NLO}} = 0.1425$, $\delta_{\text{CKM}}^{\text{NLO}} = 66.43^\circ$ (unchanged), $J_{\text{NLO}} = 2.714 \times 10^{-5}$ (PDG residual -11.8% , an $O(\lambda^4)$ problem).

Resolution of open problems. OP-P92-1: Conjecture 5.3 is false as stated; G_2 equivariance does not preserve J to $O(\lambda^2)$ under the A81 loop — it predicts $J_{\text{NLO}} = J_{\text{LO}}(1 - 4\lambda^2/5)^3$. OP-P92-2: the NLO correction to Φ is $-4\lambda^2/5$. OP-P92-3: the full $O(\lambda^2)$ NLO Wolfenstein table is given in Corollary 7.3; closing the -11.8% residual in J requires $O(\lambda^4)$ work (OP-P96-1).

References

- [1] Particle Data Group, *Review of Particle Physics*, Prog. Theor. Exp. Phys. **2024**:083C01.
- [2] L. Wolfenstein, *Parametrization of the Kobayashi-Maskawa Matrix*, Phys. Rev. Lett. **51** (1983) 1945.
- [3] C. Jarlskog, *Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Nonconservation*, Phys. Rev. Lett. **55** (1985) 1039.
- [4] M. Kobayashi and T. Maskawa, *CP-Violation in the Renormalizable Theory of Weak Interaction*, Prog. Theor. Phys. **49** (1973) 652.
- [5] R. L. Bryant, *Metrics with Exceptional Holonomy*, Ann. of Math. **126** (1987) 525–576.
- [6] K. McCrimmon, *A Taste of Jordan Algebras*, Springer-Verlag, New York, 2004.
- [7] R. D. Schafer, *An Introduction to Nonassociative Algebras*, Academic Press, New York, 1966.
- [8] J. C. Baez, *The Octonions*, Bull. Amer. Math. Soc. **39** (2002) 145–205.
- [9] L. F. Vlegels, *G_2 Subgroup Actions on $J_3(\mathbb{O})$ and the Structural Classification*, Addendum 45, TOE Corpus, May 2026.
- [10] L. F. Vlegels, *Reactor Angle and CP Phase from the G_2 Dihedral Modulus*, Addendum 66, TOE Corpus, May 2026.
- [11] L. F. Vlegels, *Bryant 3-Form, Fano Geometry, and the Origin of CP Violation*, Addendum 67, TOE Corpus, May 2026.
- [12] L. F. Vlegels, *Down-Type Quark Yukawa Matrix from $J_3(\mathbb{O})$ and the Wolfenstein A Parameter from First Principles*, Addendum 75, TOE Corpus, May 2026.

- [13] L. F. Vlegels, *$O(\lambda^2)$ Jordan Loop Correction to the Wolfenstein A Parameter and Closure of Phase $5f$* , Addendum 81, TOE Corpus, May 2026.
- [14] L. F. Vlegels, *Complete CKM Wolfenstein Parameterization from $J_3(\mathbb{O})$* , Addendum 82, TOE Corpus, May 2026.
- [15] L. F. Vlegels, *Jarlskog CP Invariant at NLO: Impact of the A_{NLO} Correction and Identification of the Remaining Open Problem*, Addendum 91, TOE Corpus, May 2026.
- [16] L. F. Vlegels, *NLO Correction to δ_{CKM} and the Jarlskog Invariant: Single-Loop Obstruction, Paired-Correction Theorem, and the G_2 Invariance Conjecture*, Addendum 92, TOE Corpus, May 2026.