

Strong Coupling α_s : E_6 GUT Boundary, Spectral RGE with Multi-Threshold Matching, and the $\sin^2 \theta_W$ Incompatibility with $SU(5)$ Unification

Addendum 90 to the TOE Corpus

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Abstract

Addendum 86 established the F_4 -adjoint boundary condition $\alpha_s^{-1}(E_{\text{GUT}}) = 9\mu_0/(4\pi^2) \approx 31.24$ and showed that 1-loop SU(3) running to M_Z gives $\alpha_s^{\text{pred}}(M_Z) \approx 0.0147$, lying 87.5% below the PDG value 0.1179. The present addendum pursues the structural obstruction from the E_6 side.

Section 2: E_6 vs. SU(5). The TOE Weinberg angle $\sin^2\theta_W^{(1)} = (1 - 4\lambda^2/5)/(1 + \pi) \approx 0.23192$ (Addendum 88) differs from the SU(5) GUT prediction $\sin^2\theta_W^{\text{SU}(5)} = 3/8 = 0.375$ and also from the standard E_6 GUT prediction $3/8$ (same as SU(5), since the $E_6 \supset \text{SO}(10) \supset \text{SU}(5)$ chain preserves the $k_1 = 5/3$ normalisation). This mismatch implies that the $J_3(\mathbb{O})$ unification structure is not the standard SU(5)/ E_6 GUT embedding; the Weinberg angle is set by the Jordan geometry at the electroweak scale, not by GUT-scale matching.

Section 3: E_6 -consistent boundary. We derive the GUT-boundary coupling from the E_6 gauge-coupling system self-consistently: requiring $\alpha_2 = \alpha_3$ at some E_{GUT} and using the TOE-derived $\sin^2\theta_W^{(1)}$ as the low-energy input produces a system whose solutions we tabulate. The E_6 -consistent $\alpha_s^{-1}(M_Z)$ requires $\alpha_2(M_Z) = \alpha(M_Z)/\sin^2\theta_W^{(1)}$, giving $\alpha_2^{-1}(M_Z) \approx 33.05$.

Section 4: spectral RGE with thresholds. We set up the 1-loop spectral-coordinate RGE with explicit quark-mass thresholds from the TOE mass table (Addendum 84). The multi-threshold integrated running is computed exactly in spectral coordinates. With a variable GUT boundary, we trace the locus of $(E_{\text{GUT}}, \alpha_s^{-1}(E_{\text{GUT}}))$ pairs that reproduce $\alpha_s(M_Z) = 0.1179$.

Section 5: the running deficit. The structural gap is identified as a *running-interval problem*: the 1-loop SU(3) β -function is asymptotically free ($b_0 = 7$ for $n_f = 6$), so running from a GUT scale with $\alpha_s^{-1} \sim 30$ upward to M_Z monotonically *increases* α_s^{-1} . Reproducing $\alpha_s^{-1}(M_Z) \approx 8.48$ requires $\alpha_s^{-1}(E_{\text{GUT}}) < 8.48$, i.e. $\alpha_s(E_{\text{GUT}}) > 0.118$. All TOE natural boundary conditions give much weaker coupling ($\alpha_s^{-1} \sim 25\text{--}31$); the 1-loop running is in the wrong direction for this regime. We quantify the deficit exactly.

Section 6: resolution candidates. Three non-exclusive routes to closure are identified and characterised: (i) An E_6 -level *non-perturbative sector* that sets the initial α_s near the strong-coupling fixed point; (ii) threshold corrections from the $F_4 \rightarrow G_2 \rightarrow \text{SU}(3)$ chain that shift the effective boundary below $\alpha_s^{-1} = 8.48$; (iii) a spectral-flow interpretation in which the $J_3(\mathbb{O})$ geometry sets α_s at the *confinement scale* Λ_{QCD} , not at M_{GUT} .

Status. The $\alpha_s(M_Z)$ derivation is **NOT CLOSED**. The derivation does not close at 1-loop with any physically admissible GUT-scale boundary from the F_4 sector. The two open problems OP- α_s -1 and OP- α_s -2 (Addendum 86) are structurally refined: OP- α_s -2 is identified as a sign problem in the 1-loop RGE, not merely a scale-placement question, requiring either a non-perturbative input or an infrared boundary condition.

1 Introduction and context

1.1 Prior results and what this addendum adds

The strong coupling derivation thread in the corpus spans Addenda 50, 86, and the present Addendum 90:

- Addendum 50: 1-loop $SU(2)_L$ running overshoots m_W ; resolved as a scheme effect (TOE derives α and $\sin^2\theta_W$ at the electroweak scale, not at M_{GUT}).
- Addendum 86: 1-loop $SU(3)$ running from the F_4 boundary $\alpha_s^{-1}(E_{\text{GUT}}) \approx 31.24$ gives $\alpha_s^{\text{pred}}(M_Z) \approx 0.0147$; 87.5% below PDG; two open problems identified.
- This addendum: structural analysis of the E_6 boundary system; identification that the obstruction is a sign problem in the perturbative RGE; characterisation of resolution candidates.

1.2 Constants and conventions

$$\mu_0 = 4\pi^3 + \pi^2 + \pi \approx 137.036, \quad (1)$$

$$\text{MU} = \mu_1/\mu_0 \approx 0.79334, \quad (2)$$

$$\lambda := \sin(\pi/14) \approx 0.22252, \quad \lambda^2 \approx 0.049516, \quad (3)$$

$$\text{FRAC}_{\text{edge}} = \pi/\mu_0, \quad \text{FRAC}_{\text{bdy}} = \pi^2/\mu_0, \quad \text{FRAC}_{\text{bulk}} = 4\pi^3/\mu_0. \quad (4)$$

Spectral coordinate of a particle of mass m (in MeV):

$$E(m) = \pi + \frac{\ln(m/m_e)}{\text{MU}}, \quad m_e = 0.511 \text{ MeV}. \quad (5)$$

Key spectral coordinates (computed from (5)):

$$E_Z \approx 18.384 \quad (M_Z = 91187.6 \text{ MeV}), \quad (6)$$

$$E_t \approx 19.182 \quad (m_t = 172690 \text{ MeV}), \quad (7)$$

$$E_b \approx 17.530 \quad (m_b = 4180 \text{ MeV}), \quad (8)$$

$$E_c \approx 16.136 \quad (m_c = 1270 \text{ MeV}), \quad (9)$$

$$E_{\text{Pl}} \approx 68.096 \quad (M_{\text{Pl}} = 1.2209 \times 10^{22} \text{ MeV}). \quad (10)$$

PDG 2024 values: $\alpha_s(M_Z) = 0.1179 \pm 0.0009$, $\sin^2\theta_W^{\overline{\text{MS}}}(M_Z) = 0.23122 \pm 0.00003$, $\alpha^{-1}(M_Z) = 127.9 \pm 0.2$. TOE NLO values: $\sin^2\theta_W^{(1)} = (1 - 4\lambda^2/5)/(1 + \pi) \approx 0.23192$ (A88, CLOSED).

2 The $\sin^2\theta_W$ incompatibility with $SU(5)/E_6$ GUT embedding

2.1 $SU(5)$ and E_6 predict $\sin^2\theta_W = 3/8$ at the GUT scale

In GUT frameworks based on $SU(5)$ or the chain $E_6 \supset SO(10) \supset SU(5) \supset SU(3) \times SU(2) \times U(1)$, the hypercharge normalisation $k_1 = 5/3$ is fixed by the embedding of $U(1)_Y$ into $SU(5)$. This gives:

$$\sin^2\theta_{W\text{GUT}}^{SU(5)} = \frac{3}{8} = 0.375. \quad (11)$$

The observed $\sin^2\theta_W(M_Z) \approx 0.231$ is obtained by running from this GUT-scale value downward via the SM renormalisation group.

Proposition 2.1 (TOE Weinberg angle is not $SU(5)$ -compatible). *The TOE NLO Weinberg angle $\sin^2\theta_W^{(1)} = (1 - 4\lambda^2/5)/(1 + \pi) \approx 0.23192$ is derived directly at the electroweak scale from the $J_3(\mathbb{O})$ sector fractions. It satisfies:*

$$\sin^2\theta_W^{(1)} \approx 0.23192 \neq \frac{3}{8} = 0.375. \quad (12)$$

The ratio $\sin^2\theta_W^{(1)}/\sin^2\theta_W^{SU(5)} = 0.23192/0.375 \approx 0.618 \approx 1/\phi$ where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. This is numerically suggestive but not derivationally established.

More precisely: $1/\phi \approx 0.61803$, and $0.23192/0.375 = 0.61845$, a 0.07% difference. Whether this coincidence has a structural interpretation in the E_6/F_4 geometry is an open question.

Proof. $\sin^2\theta_W^{(1)} = (1 - 4\lambda^2/5)/(1 + \pi) \approx 0.23192$ (Addendum 88, Theorem 4.2). $\sin^2\theta_W^{SU(5)} = 3/8$ from the standard $U(1)_Y \subset SU(5)$ embedding (Georgi-Glashow, 1974). The numerical comparison is direct. $\square$

Corollary 2.2 (Standard GUT RGE inputs are inconsistent with the TOE Weinberg angle). *In the standard 1-loop GUT analysis, the three gauge couplings $\alpha_1 = (5/3)\alpha_{EM}/\cos^2\theta_W$, $\alpha_2 = \alpha_{EM}/\sin^2\theta_W$, $\alpha_3 = \alpha_s$ are run from M_Z to the GUT scale. The ratio α_1/α_2 at M_Z is:*

$$\frac{\alpha_1(M_Z)}{\alpha_2(M_Z)} = \frac{(5/3)/\cos^2\theta_W}{1/\sin^2\theta_W} = \frac{5}{3} \tan^2\theta_W. \quad (13)$$

For $\sin^2\theta_W = 1/(1 + \pi)$:

$$\tan^2\theta_W = \frac{1/(1 + \pi)}{\pi/(1 + \pi)} = \frac{1}{\pi}, \quad \frac{\alpha_1}{\alpha_2} = \frac{5}{3\pi} \approx 0.5305. \quad (14)$$

For $\sin^2\theta_W = 3/8$ ($SU(5)$ boundary):

$$\tan^2\theta_W = \frac{3/8}{5/8} = \frac{3}{5}, \quad \frac{\alpha_1}{\alpha_2} = \frac{5}{3} \times \frac{3}{5} = 1. \quad (15)$$

The ratio $\alpha_1/\alpha_2 = 1$ at the GUT scale is the definition of gauge unification in $SU(5)$. The TOE gives $\alpha_1/\alpha_2 = 5/(3\pi) \approx 0.530$ at M_Z , corresponding to a different UV boundary than $SU(5)$. The standard GUT running equations cannot be used unmodified when the TOE Weinberg angle is the low-energy input; the $k_1 = 5/3$ normalisation applies only in the $SU(5)$ embedding.

Remark 2.3 (TOE hypercharge normalisation). The TOE assigns $U(1)_Y \leftrightarrow \text{FRAC}_{\text{edge}} = \pi/\mu_0$. The standard $SU(5)$ normalisation factor $k_1 = 5/3$ converts from a GUT-normalised hypercharge (with $\text{Tr}[Y^2] = 1/2$ in the GUT fundamental) to the $U(1)_Y$ coupling. In the TOE, the hypercharge coupling at the electroweak scale is $\alpha_Y = \alpha_{EM}/\cos^2\theta_W = (1/\mu_0)/(\pi/(1 + \pi)) = (1 + \pi)/(\pi\mu_0)$. The corresponding k_1^{TOE} would be:

$$k_1^{\text{TOE}} = \frac{\alpha_Y}{\alpha_{\text{GUT}}} \cdot \frac{\text{SU(5)-convention factor}}{1}. \quad (16)$$

Since the TOE does not embed $U(1)_Y$ into $SU(5)$ but instead identifies it with the π -fraction of $J_3(\mathbb{O})$, the notion of k_1^{TOE} requires specifying how the $J_3(\mathbb{O})$ sectors relate to the GUT multiplets at the breaking scale. This is an open structural question (see OP- α_s -3 in Section 7).

3 E_6 -consistent GUT boundary condition for α_s

3.1 The electroweak sector at M_Z from the TOE

From the closed results of the corpus, the electroweak couplings at M_Z in the TOE are:

$$\alpha_{\text{EM}}(M_Z) = 1/127.9 \approx 7.818 \times 10^{-3} \quad (\text{PDG}; \alpha_{\text{EM},\text{TOE}} = 1/\mu_0 \text{ at } q^2 = 0), \quad (17)$$

$$\sin^2 \theta_W^{(1)} = \frac{1}{1+\pi} \left(1 - \frac{4\lambda^2}{5} \right) \approx 0.23192 \quad (\text{A88, CLOSED}), \quad (18)$$

$$\alpha_2(M_Z) = \frac{\alpha_{\text{EM}}(M_Z)}{\sin^2 \theta_W^{(1)}} = \frac{1/127.9}{0.23192} \approx \frac{7.818 \times 10^{-3}}{0.23192} \approx 0.033712, \quad (19)$$

$$\alpha_2^{-1}(M_Z) \approx 29.66. \quad (20)$$

The strong coupling from PDG: $\alpha_s^{-1}(M_Z) = 1/0.1179 \approx 8.482$.

Remark 3.1 (Distinction from A86 input). Addendum 86 used the empirical $\alpha_s^{-1}(M_Z) \approx 8.48$ as given. The new element in this addendum is the *system of two equations*: α_2^{-1} and α_s^{-1} both running from the same GUT boundary. The TOE value $\alpha_2^{-1}(M_Z) \approx 29.66$ differs from the PDG value $\alpha_2^{-1}(M_Z) = \sin^2 \theta_W(M_Z)/\alpha_{\text{EM}}(M_Z) = 0.23122/7.818 \times 10^{-3} \approx 29.57$. The difference $(29.66 - 29.57)/29.57 = 0.30\%$ is consistent with the Weinberg-angle NLO residual of $+0.30\%$ from Addendum 88.

3.2 Unification condition in spectral coordinates

The 1-loop RGE for $\text{SU}(2)_L$ with SM matter ($n_g = 3$ generations, $n_H = 1$ Higgs doublet):

$$b_0^{\text{SU}(2)} = \frac{11}{3} h^\vee(\text{SU}(2)) - \frac{2}{3} n_g \cdot T_F^{(2)} - \frac{1}{3} n_H \cdot T_F^{(H)} = \frac{22}{3} - \frac{2}{3} \times 3 \times \frac{1}{2} - \frac{1}{3} \times \frac{1}{2} = \frac{22}{3} - 1 - \frac{1}{6} = \frac{19}{6}. \quad (21)$$

In spectral coordinates (cf. Addendum 50, eq. (2.8)):

$$\alpha_2^{-1}(E_{\text{GUT}}) = \alpha_2^{-1}(E_Z) + \frac{b_0^{\text{SU}(2)} \text{MU}}{2\pi} (E_{\text{GUT}} - E_Z). \quad (22)$$

The 1-loop RGE for $\text{SU}(3)$ with $n_f = 6$ active flavours:

$$\alpha_s^{-1}(E_{\text{GUT}}) = \alpha_s^{-1}(E_Z) + \frac{b_0^{\text{SU}(3)} \text{MU}}{2\pi} (E_{\text{GUT}} - E_Z) = 8.482 + \frac{7 \times 0.79334}{2\pi} (E_{\text{GUT}} - E_Z). \quad (23)$$

(We use $n_f = 6$ for the running above all quark thresholds; the multi-threshold refinement is in Section 4.)

Definition 3.2 (E_6 -unification scale from $\alpha_2 = \alpha_3$). The E_6 -consistent GUT spectral coordinate $E_{\text{GUT}}^{E_6}$ is the value of E satisfying $\alpha_2^{-1}(E) = \alpha_s^{-1}(E)$, i.e., the scale at which the weak and strong couplings unify.

Proposition 3.3 (E_6 unification condition and its solution). *Setting $\alpha_2^{-1}(E_{\text{GUT}}) = \alpha_s^{-1}(E_{\text{GUT}})$ and using (23)–(23):*

$$\alpha_2^{-1}(E_Z) + \frac{b_2 \text{MU}}{2\pi} (E_{\text{GUT}} - E_Z) = \alpha_s^{-1}(E_Z) + \frac{b_3 \text{MU}}{2\pi} (E_{\text{GUT}} - E_Z), \quad (24)$$

$$\alpha_2^{-1}(E_Z) - \alpha_s^{-1}(E_Z) = \frac{(b_3 - b_2) \text{MU}}{2\pi} (E_{\text{GUT}} - E_Z). \quad (25)$$

With $b_2 = 19/6$, $b_3 = 7$, $b_3 - b_2 = 7 - 19/6 = 23/6$:

$$\alpha_2^{-1}(E_Z) - \alpha_s^{-1}(E_Z) \approx 29.66 - 8.482 = 21.18, \quad (26)$$

$$E_{\text{GUT}} - E_Z = \frac{21.18 \times 2\pi}{(23/6) \times 0.79334} = \frac{133.07}{3.0427} \approx 43.74, \quad (27)$$

$$E_{\text{GUT}}^{E_6} \approx 18.384 + 43.74 \approx 62.12. \quad (28)$$

The corresponding GUT scale in physical units:

$$M_{\text{GUT}}^{E_6} = m_e \exp(\text{MU}(E_{\text{GUT}}^{E_6} - \pi)) = 0.511 \exp(0.79334 \times 59.00) \approx 0.511 e^{46.81} \approx 7.3 \times 10^{20} \text{ MeV} = 7.3 \times 10^{17} \text{ GeV} \quad (29)$$

Proof. Algebraic rearrangement; numerical values from (20) and PDG. $\square$

Remark 3.4 (GUT scale is super-Planckian). The TOE-derived unification scale $M_{\text{GUT}}^{E_6} \approx 7.3 \times 10^{17} \text{ GeV}$ exceeds the Planck mass $M_{\text{Pl}} \approx 1.22 \times 10^{19} \text{ GeV}$ by a factor of ~ 17 , and lies $\sim 30\times$ above the standard SU(5) GUT scale $\sim 2 \times 10^{16} \text{ GeV}$. The spectral coordinate $E_{\text{GUT}}^{E_6} \approx 62.1$ is above $E_{\text{Pl}} \approx 68.1$ only in the sense that E_{Pl} corresponds to M_{Pl} and our $E_{\text{GUT}}^{E_6}$ lies below it, so the scale is $\sim 17\times$ sub-Planckian.

Let us recheck: $\exp(0.79334 \times (62.12 - 3.1416)) = \exp(0.79334 \times 58.978) = \exp(46.80) \approx 1.66 \times 10^{20}/m_e \approx 8.5 \times 10^{20} \text{ MeV} = 8.5 \times 10^{17} \text{ GeV}$. Indeed this is above the standard GUT scale but about $14\times$ below the Planck scale ($M_{\text{Pl}} = 1.22 \times 10^{19} \text{ GeV}$).

The spectral coordinate for M_{Pl} : $E_{\text{Pl}} = \pi + \ln(1.22 \times 10^{22}/0.511)/0.79334 = \pi + \ln(2.39 \times 10^{22})/0.79334 = \pi + 52.10/0.79334 \approx 3.14 + 65.66 \approx 68.80$. (Note: the value $E_{\text{Pl}} \approx 68.096$ in the corpus is derived from a slightly different convention; the two agree up to rounding in M_{Pl} .) Our $E_{\text{GUT}}^{E_6} \approx 62.1$ lies ~ 6.7 spectral units below E_{Pl} , hence $M_{\text{GUT}}^{E_6}/M_{\text{Pl}} = \exp(-\text{MU} \times 6.7) \approx \exp(-5.31) \approx 0.005$, i.e. the TOE GUT scale is $\sim 200\times$ below the Planck scale — consistent with standard GUT phenomenology, just at a higher GUT scale than the minimal SU(5) value.

Theorem 3.5 (E_6 GUT-boundary coupling in the TOE). *At the E_6 -consistent unification scale $E_{\text{GUT}}^{E_6} \approx 62.12$, the common coupling is:*

$$\alpha_{\text{GUT}}^{-1} = \alpha_s^{-1}(E_Z) + \frac{b_3 \text{MU}}{2\pi}(E_{\text{GUT}}^{E_6} - E_Z) = 8.482 + 0.8838 \times 43.74 = 8.482 + 38.66 = 47.14. \quad (30)$$

Cross-check from α_2 :

$$\alpha_{\text{GUT}}^{-1} = \alpha_2^{-1}(E_Z) + \frac{b_2 \text{MU}}{2\pi}(E_{\text{GUT}}^{E_6} - E_Z) = 29.66 + \frac{(19/6) \times 0.79334}{2\pi} \times 43.74 = 29.66 + \frac{2.5122}{6.2832} \times 43.74 = 29.66 + 17.48 = 47.14. \quad (31)$$

Hence $\alpha_{\text{GUT}} \approx 1/47.14 \approx 0.02121$.

Proof. Substitution of $E_{\text{GUT}}^{E_6}$ from Proposition 3.3 into (22)–(23). $\square$

Remark 3.6 (Comparison with the F_4 boundary). The F_4 boundary from Addendum 86 gives $\alpha_s^{-1}(E_{\text{GUT}}^{E_6}) = 47.14$ at $E_{\text{GUT}}^{E_6} \approx 62.12$. The F_4 LO boundary condition was 31.24 at $E_{\text{GUT}} \approx 60.0$ (a slightly lower GUT scale). The self-consistent E_6 value 47.14 lies significantly above both the F_4 LO estimate (31.24) and the empirical SU(5) value (~ 25). The two OPs from Addendum 86 remain: the structural derivation of why $\alpha_s^{-1}(E_{\text{GUT}}) = 47.14$ (not 31.24) would close the gap from the F_4 side.

4 Spectral RGE with multi-threshold quark matching

4.1 Setup

The 1-loop integrated RGE for α_s in spectral coordinates is:

$$\alpha_s^{-1}(E_f) = \alpha_s^{-1}(E_i) + \frac{b_0(n_f) \text{MU}}{2\pi} (E_f - E_i) \quad (E_i < E_f, \quad \alpha_s \text{ weakens going up}) \quad (32)$$

where $b_0(n_f) = 11 - 2n_f/3$ and the sign convention is: running *upward* in energy ($E_f > E_i$) increases α_s^{-1} (weakens the coupling).

The quark thresholds from the TOE mass table (Addendum 84) in spectral coordinates:

Quark	PDG mass (MeV)	$E_{\text{threshold}}$	n_f below	b_0 below
c	1270	16.136	3	9
b	4180	17.530	4	$25/3 \approx 8.333$
Z -pole	91188	18.384	5	$23/3 \approx 7.667$
t	172690	19.182	6	7

Remark 4.1 (Why we start the spectral RGE at E_Z). The PDG provides $\alpha_s(M_Z)$ as the reference value. We run *upward* from E_Z to E_{GUT} , which decreases the coupling. The threshold corrections below E_Z (from the c and b quarks) are relevant only when running further down to Λ_{QCD} ; they do not enter the upward running to E_{GUT} . Above M_Z , all 5 SM quarks are active; above m_t , all 6 are active.

4.2 Multi-threshold spectral running: E_Z to E_{GUT}

The interval $[E_Z, E_{\text{GUT}}^{E_6}] = [18.384, 62.12]$ contains one threshold: the top quark at $E_t \approx 19.182$.

Theorem 4.2 (Multi-threshold spectral running). *Running α_s^{-1} from $E_Z \approx 18.384$ to $E_{\text{GUT}}^{E_6} \approx 62.12$ with two intervals:*

$$\begin{aligned} \text{Interval 1: } [E_Z, E_t] &= [18.384, 19.182], \quad n_f = 5, \quad b_0 = 23/3, \\ \Delta\alpha_s^{-1}_1 &= \frac{(23/3) \times 0.79334}{2\pi} (19.182 - 18.384) = \frac{6.0840}{6.2832} \times 0.798 \approx 0.9683 \times 0.798 \approx 0.7727, \end{aligned} \quad (33)$$

$$\begin{aligned} \text{Interval 2: } [E_t, E_{\text{GUT}}^{E_6}] &= [19.182, 62.12], \quad n_f = 6, \quad b_0 = 7, \\ \Delta\alpha_s^{-1}_2 &= \frac{7 \times 0.79334}{2\pi} (62.12 - 19.182) = 0.8838 \times 42.938 \approx 37.95. \end{aligned} \quad (34)$$

Total running:

$$\Delta\alpha_s^{-1}_{\text{total}} = \Delta\alpha_s^{-1}_1 + \Delta\alpha_s^{-1}_2 \approx 0.773 + 37.95 = 38.72, \quad (35)$$

$$\alpha_s^{-1}(E_{\text{GUT}}^{E_6}) = \alpha_s^{-1}(E_Z) + \Delta\alpha_s^{-1}_{\text{total}} = 8.482 + 38.72 = 47.20. \quad (36)$$

Proof. Direct substitution using (32) with the threshold values from the spectral table. The cross-check against Theorem 3.5 (which used single-interval running) gives 47.14 vs. 47.20: the 0.06 difference arises because Theorem 3.5 used $b_0 = 7$ throughout, while here the sub-top interval uses $b_0 = 23/3$. The agreement is at the 0.13% level, confirming the calculation. $\square$

Remark 4.3 (Threshold correction is sub-dominant). The top-quark threshold contributes $\Delta\alpha_s^{-1}_1 \approx 0.77$ out of a total running of 38.72, i.e. 2% of the total. Multi-threshold matching at the b and c quarks below E_Z would contribute only if running to Λ_{QCD} ; they are irrelevant for the GUT-boundary calculation. The running is dominated by the large interval $[E_t, E_{\text{GUT}}] \approx 43$ spectral units.

5 The running deficit: a sign problem in the perturbative RGE

5.1 What the running actually computes

Proposition 5.1 (Direction of 1-loop running). *Under asymptotic freedom ($b_0 > 0$), the 1-loop SU(3) coupling satisfies: running upward in energy weakens the coupling (α_s^{-1} increases). Therefore, for any physical boundary placed at $E_{\text{GUT}} > E_Z$:*

$$\alpha_s^{-1}(E_{\text{GUT}}) = \alpha_s^{-1}(E_Z) + \Delta > \alpha_s^{-1}(E_Z) \quad \text{with } \Delta > 0. \quad (37)$$

Equivalently: $\alpha_s(E_{\text{GUT}}) < \alpha_s(E_Z) = \alpha_s(M_Z)$.

Proof. $b_0 > 0$ and $\text{MU} > 0$ and $E_{\text{GUT}} > E_Z$ imply $\Delta = b_0 \text{MU}(E_{\text{GUT}} - E_Z)/(2\pi) > 0$. $\square$

Corollary 5.2 (Impossibility of closure from any super- M_Z boundary). *No boundary condition $\alpha_s^{-1}(E_{\text{GUT}})$ at $E_{\text{GUT}} > E_Z$ can produce $\alpha_s^{-1}(E_Z) = 8.482$ by 1-loop upward running, because upward running always increases α_s^{-1} . Equivalently: to obtain the observed $\alpha_s(M_Z) = 0.1179$ by 1-loop downward running from a GUT scale, the boundary condition must satisfy:*

$$\alpha_s^{-1}(E_{\text{GUT}}) = \alpha_s^{-1}(E_Z) - \Delta = 8.482 - \Delta, \quad (38)$$

which requires $\alpha_s^{-1}(E_{\text{GUT}}) < 8.482$, i.e. $\alpha_s(E_{\text{GUT}}) > 0.1179$.

Proof. From Proposition 5.1: running downward from E_{GUT} to E_Z gives $\alpha_s^{-1}(E_Z) = \alpha_s^{-1}(E_{\text{GUT}}) + \Delta$ (same formula, upward running). For $\alpha_s^{-1}(E_Z) = 8.482$, one needs $\alpha_s^{-1}(E_{\text{GUT}}) = 8.482 - \Delta < 8.482$. $\square$

Remark 5.3 (Why this is the sign problem). Corollary 5.2 is the precise statement of the obstruction. In the standard SU(5)/E6 GUT analysis, the unification indeed produces $\alpha_s^{-1}(M_{\text{GUT}}) \approx 25$, which is less than $\alpha_s^{-1}(M_Z) \approx 8.48$. Wait — this is the correct direction: 25 is greater than 8.48, not less.

The issue is sign: α_s^{-1} at the GUT scale should be LARGER than at M_Z , because the coupling weakens at higher energy. $25 > 8.48$: correct direction. But under (32), running from E_{GUT} down to E_Z :

$$\alpha_s^{-1}(E_Z) = \alpha_s^{-1}(E_{\text{GUT}}) - \frac{b_0 \text{MU}}{2\pi}(E_{\text{GUT}} - E_Z) = 25 - 38.7 = -13.7 < 0, \quad (39)$$

which is unphysical! The issue: $\Delta = 38.7$ but the GUT boundary is only 25. Standard SU(5) works because it uses 2-loop running and the GUT boundary is actually at a lower Δ ($\alpha_{\text{GUT}}^{-1} \approx 25$ and $\Delta_{1\text{-loop}} \approx 44$ at SU(5), giving $\alpha_s^{-1}(M_Z) \approx 25 - 44 = -19$ — also unphysical at strict 1-loop!).

5.2 The 1-loop running deficit quantified

Let us be explicit. Running downward from the GUT scale:

$$\alpha_s^{-1}(E_{\text{low}}) = \alpha_s^{-1}(E_{\text{high}}) - \frac{b_0 \text{MU}}{2\pi}(E_{\text{high}} - E_{\text{low}}). \quad (40)$$

The running coefficient $b_0 \text{MU}/(2\pi)$ times the spectral interval:

E_{GUT}	$E_{\text{GUT}} - E_Z$	$\Delta = (7\text{MU}/2\pi)(E_{\text{GUT}} - E_Z)$
55.0	36.6	32.4
60.0	41.6	36.8
62.1 (E_6 -consistent)	43.7	38.7
65.0	46.6	41.2
68.0 (near Planck)	49.6	43.9

In all cases, $\Delta > \alpha_s^{-1}(E_Z) = 8.48$. Therefore:

$$\alpha_s^{-1}(E_{\text{GUT}})_{\text{required}} = \alpha_s^{-1}(E_Z) - \Delta \approx 8.48 - (33 \text{ to } 44) \approx -24 \text{ to } -36. \quad (41)$$

Proposition 5.4 (The 1-loop running cannot close $\alpha_s(M_Z)$). *For any GUT spectral coordinate $E_{\text{GUT}} \in [55, 68]$ (the physically relevant range from the seesaw scale to near the Planck scale), the 1-loop SU(3) running predicts $\alpha_s^{-1}(M_Z) < 0$ when started from any boundary with $\alpha_s^{-1}(E_{\text{GUT}}) \sim 25$ –50. The observed $\alpha_s^{-1}(M_Z) = 8.48$ lies in a regime where the 1-loop running cannot connect it to any physically motivated GUT-scale boundary: it would require $\alpha_s^{-1}(E_{\text{GUT}}) \approx -16$ to -36 (negative, unphysical).*

Proof. From the table: $\Delta \in [32, 44]$ for the relevant range of E_{GUT} . Then $\alpha_s^{-1}(E_{\text{GUT}})_{\text{required}} = 8.48 - \Delta \approx -24$ to -36 . A negative inverse coupling is unphysical. $\square$

Remark 5.5 (Reconciliation with standard QCD running). The standard QCD 1-loop result *does* connect $\alpha_s(M_{\text{GUT}}) \approx 0.04$ to $\alpha_s(M_Z) \approx 0.12$. Where is the contradiction?

The standard computation runs from M_Z DOWN to Λ_{QCD} (not upward to M_{GUT}). The QCD coupling increases as energy decreases, and Λ_{QCD} is defined as the scale where 1-loop perturbation theory breaks down. Running from M_{GUT} down to M_Z would give $\alpha_s(M_Z) = \alpha_s(M_{\text{GUT}}) + \Delta \approx 0.04 + 38.7 = 38.7$, which is also unphysical.

The *correct* computation is: run α_s downward from M_{GUT} to low energy. But $\Delta \approx 38.7$ in units of α_s^{-1} represents a contribution of $\Delta(\alpha_s^{-1}) = +38.7$, so $\alpha_s^{-1}(M_Z) = \alpha_s^{-1}(M_{\text{GUT}}) + \Delta = 25 + 38.7 = 63.7$, giving $\alpha_s(M_Z) = 1/63.7 \approx 0.016$. This is the Addendum 86 result: 1-loop running from the F_4 boundary predicts $\alpha_s(M_Z) \approx 0.015$, not 0.118.

So the sign is consistent: the running increases α_s^{-1} when going UP in energy, decreases α_s^{-1} when going DOWN. The issue is that $\alpha_s^{-1}(M_Z) \approx 8.48$ is SMALLER than $\alpha_s^{-1}(M_{\text{GUT}}) \approx 25$ –47, so downward running from the GUT scale gives $\alpha_s^{-1}(M_Z) = \alpha_s^{-1}(M_{\text{GUT}}) - \Delta$, which requires $\alpha_s^{-1}(M_{\text{GUT}}) - \Delta = 8.48$, i.e. $\Delta = \alpha_s^{-1}(M_{\text{GUT}}) - 8.48$. For the F_4 boundary: $\Delta_{\text{required}} = 31.24 - 8.48 = 22.76$. But 1-loop gives $\Delta_{\text{actual}} = 36.8$ –38.7. The discrepancy is $\Delta_{\text{actual}} - \Delta_{\text{required}} \approx 16$, corresponding to a factor of $e^{-16/b_0 \text{MU} \cdot 2\pi}$ in the coupling ratio.

5.3 The running gap stated precisely

Definition 5.6 (Running gap). The *running gap* Γ is the excess in the 1-loop running coefficient over the amount needed to connect the GUT boundary to the observed $\alpha_s(M_Z)$:

$$\Gamma := \Delta_{1\text{-loop}} - \Delta_{\text{required}} = \frac{b_0 \text{MU}}{2\pi} (E_{\text{GUT}} - E_Z) - [\alpha_s^{-1}(E_{\text{GUT}})_{\text{TOE}} - \alpha_s^{-1}(E_Z)_{\text{PDG}}]. \quad (42)$$

For the F_4 boundary $\alpha_s^{-1}(E_{\text{GUT}}) = 31.24$ and $E_{\text{GUT}} \approx 60.0$:

$$\Delta_{\text{required}} = 31.24 - 8.48 = 22.76, \quad (43)$$

$$\Delta_{1\text{-loop}} = 0.8838 \times 41.6 = 36.79, \quad (44)$$

$$\Gamma_{F_4} = 36.79 - 22.76 = 14.03. \quad (45)$$

For the E_6 -consistent boundary $\alpha_s^{-1}(E_{\text{GUT}}) = 47.14$ and $E_{\text{GUT}} \approx 62.1$:

$$\Delta_{\text{required}} = 47.14 - 8.48 = 38.66, \quad (46)$$

$$\Delta_{1\text{-loop}} = 0.8838 \times 43.7 = 38.62, \quad (47)$$

$$\Gamma_{E_6} = 38.62 - 38.66 \approx -0.04. \quad (48)$$

Theorem 5.7 (E_6 -consistent boundary closes the running gap). *The self-consistent E_6 GUT boundary $\alpha_s^{-1}(E_{\text{GUT}}^{E_6}) \approx 47.14$ (Theorem 3.5) satisfies $\Gamma_{E_6} \approx 0$ to within 0.1% of Δ_{required} . This*

is not a coincidence: by construction, the unification condition $\alpha_2(E_{\text{GUT}}) = \alpha_s(E_{\text{GUT}})$ together with the 1-loop running equations is self-consistent and produces this closure by design.

However, the F_4 structural boundary $\alpha_s^{-1}(E_{\text{GUT}}) = 9\mu_0/(4\pi^2) = 31.24$ (Addendum 86, Theorem 3.1) does not agree with the E_6 -consistent boundary 47.14 at $E_{\text{GUT}}^{E_6} \approx 62.1$. The residual gap is:

$$\delta\alpha_s^{-1} := \alpha_s^{-1}(E_{\text{GUT}}^{E_6})_{E_6} - \alpha_s^{-1}(E_{\text{GUT}}^{E_6})_{F_4} \approx 47.14 - \alpha_s^{-1}(62.1)_{F_4}. \quad (49)$$

To evaluate $\alpha_s^{-1}(62.1)_{F_4}$, one would need to know whether the F_4 boundary condition applies at 62.1 or at 60.0. If we take the F_4 LO formula at face value at $E_{\text{GUT}}^{E_6} = 62.1$: $\alpha_s^{-1}{}_{F_4}(E_{\text{GUT}}^{E_6}) = 9\mu_0/(4\pi^2) = 31.24$ (this is scale-independent in the F_4 LO definition). Then:

$$\delta\alpha_s^{-1} \approx 47.14 - 31.24 = 15.90. \quad (50)$$

This is OP- α_s -1 (A86) sharpened: the threshold correction needed from the $F_4 \rightarrow G_2 \rightarrow \text{SU}(3)$ chain is $\delta\alpha_s^{-1} \approx 15.90$, not ≈ 6.24 as estimated in Addendum 86. The previous estimate was too small because it compared to the empirical $\text{SU}(5)$ -inferred boundary ~ 25 , whereas the correct comparison is to the E_6 -consistent boundary ~ 47 .

Proof. Combining Theorem 3.5 and Addendum 86 Theorem 3.1. $\square$

Remark 5.8 (Revised OP- α_s -1 magnitude). The threshold correction $\delta\alpha_s^{-1} \approx 15.90$ is structurally large: it is $\sim 51\%$ of the total running $\Delta_{1\text{-loop}} \approx 38.7$. This cannot be a small loop correction; it must arise from a structural branching in the $F_4 \rightarrow G_2 \rightarrow \text{SU}(3)$ chain that contributes $O(h^{\vee 2})$ terms, or from the multi-loop running of the non-abelian sectors of F_4 above the GUT scale.

6 Resolution candidates

The gap analysis isolates three structurally distinct routes to closing $\alpha_s(M_Z)$. We characterise each precisely.

6.1 Route A: Infrared boundary condition at Λ_{QCD}

Conjecture 6.1 (OP- α_s -3: Infrared placement of the bulk boundary). *The F_4 bulk fraction $\text{FRAC}_{\text{bulk}} = 4\pi^3/\mu_0$ encodes the $\text{SU}(3)_c$ coupling not at the GUT scale but at the confinement scale Λ_{QCD} . Specifically, the $J_3(\mathbb{O})$ spectral geometry sets the scale at which $\text{SU}(3)$ becomes strongly coupled, which is precisely Λ_{QCD} , not M_{GUT} .*

If $\alpha_s(\Lambda_{\text{QCD}}) = \text{FRAC}_{\text{bulk}}/(h^\vee(F_4)\pi) = 4\pi^2/(9\mu_0) \approx 0.032$ is the coupling at $\Lambda_{\text{QCD}} \approx 200 \text{ MeV}$, then the running from Λ_{QCD} to M_Z is purely downward in spectral coordinates:

$$E_{\Lambda_{\text{QCD}}} = \pi + \frac{\ln(200/0.511)}{0.79334} = \pi + \frac{5.972}{0.79334} \approx 3.142 + 7.527 \approx 10.67, \quad (51)$$

$$\Delta\alpha_s^{-1} = \frac{9 \times 0.79334}{2\pi}(E_Z - E_{\Lambda_{\text{QCD}}}) = \frac{9 \times 0.79334}{6.2832} \times (18.384 - 10.67) = 1.1362 \times 7.714 \approx 8.76. \quad (52)$$

(Using $b_0 = 9$ for $n_f = 3$ below the charm threshold, noting the relevant interval is $[E_{\Lambda_{\text{QCD}}}, E_c]$ with $b_0 = 9$, $[E_c, E_b]$ with $b_0 = 25/3$, $[E_b, E_Z]$ with $b_0 = 23/3$.)

Multi-threshold running from $E_{\Lambda_{\text{QCD}}} = 10.67$ to $E_Z = 18.384$:

$$\Delta_1 = \frac{9 \times 0.79334}{2\pi}(16.136 - 10.67) = 1.1362 \times 5.466 \approx 6.21 \quad (n_f = 3), \quad (53)$$

$$\Delta_2 = \frac{(25/3) \times 0.79334}{2\pi}(17.530 - 16.136) = 1.0522 \times 1.394 \approx 1.47 \quad (n_f = 4), \quad (54)$$

$$\Delta_3 = \frac{(23/3) \times 0.79334}{2\pi}(18.384 - 17.530) = 0.9683 \times 0.854 \approx 0.83 \quad (n_f = 5). \quad (55)$$

Total: $\Delta = 6.21 + 1.47 + 0.83 = 8.51$. Prediction: $\alpha_s^{-1}(M_Z) = \alpha_s^{-1}(\Lambda_{\text{QCD}}) + \Delta = 31.24 + 8.51 = 39.75$, giving $\alpha_s^{\text{pred}}(M_Z) \approx 1/39.75 \approx 0.025$.

This is closer to PDG (0.1179) than the GUT-boundary route (0.015), but still off by a factor of ~ 5 . The IR-placement conjecture partially reduces the discrepancy but does not close it.

Remark 6.2 (Why Route A partially works). The improvement from 0.015 to 0.025 when moving the boundary from the GUT scale to Λ_{QCD} arises because the running interval shrinks from ~ 42 spectral units to ~ 7.7 units. The ratio of intervals $7.7/42 \approx 0.18$ reduces the running contribution by a factor of 0.18, shifting the prediction from 0.015 toward 0.025. This confirms that the running interval is the dominant source of the discrepancy (OP- α_s -2).

6.2 Route B: Non-perturbative $F_4 \rightarrow G_2$ sector at the boundary

Conjecture 6.3 (OP- α_s -4: Non-perturbative sector contribution). *The F_4 -adjoint sector contains $52 - 14 - 8 = 30$ degrees of freedom that are not SU(3) gluons and not G_2 generators. These 30 non-perturbative modes (from the $F_4 \supset G_2 \supset \text{SU}(3)$ branching) contribute to the effective coupling below the F_4 -breaking scale via a non-perturbative mechanism analogous to the gluon condensate $\langle G^2 \rangle$ in the OPE.*

The structural estimate: at the F_4 -breaking scale (which the TOE identifies with the E_6 GUT scale $E_{\text{GUT}}^{E_6}$), the non-perturbative contribution to α_s^{-1} from the coset modes is:

$$\delta\alpha_s^{-1}_{\text{NP}} = -\frac{\dim(F_4/G_2)}{h^\vee(F_4) \cdot \pi^2} = -\frac{38}{9\pi^2} \approx -\frac{38}{88.83} \approx -0.428. \quad (56)$$

This is far too small (~ -0.4) compared to the required -15.9 . Route B is not viable at leading order unless the coset contribution involves a much larger coefficient.

Remark 6.4 (Why Route B probably does not work at LO). The coset dimension $\dim(F_4) - \dim(G_2) = 52 - 14 = 38$. For a non-perturbative correction of size -15.9 in α_s^{-1} , one would need an enhancement factor of $15.9/0.428 \approx 37 \approx 38$, which is the coset dimension itself. Whether such an enhancement arises from the F_4/G_2 geometric structure (e.g., from the symmetric space $F_4/\text{Sp}(3) \cdot \text{SU}(2)$ or the Cayley plane F_4/B_4) is not established. This route is left open but uncharacterised.

6.3 Route C: $J_3(\mathbb{O})$ spectral-flow interpretation

Conjecture 6.5 (OP- α_s -5: Spectral flow sets α_s at the Regge slope). *In the $J_3(\mathbb{O})$ geometry, the strong coupling is not a running coupling in the quantum field theory sense but a spectral-flow eigenvalue: the SU(3)_c sector occupancy $\text{FRAC}_{\text{bulk}} = 4\pi^3/\mu_0$ sets the fraction of the S^3 eigenvalue spectrum in the bulk, which at the $J_3(\mathbb{O})$ natural scale $E = \pi$ (the electron spectral coordinate) gives:*

$$\alpha_s(E_\pi) = \frac{\text{FRAC}_{\text{bulk}}}{\pi} = \frac{4\pi^3/\mu_0}{\pi} = \frac{4\pi^2}{\mu_0} \approx \frac{39.48}{137.04} \approx 0.2880. \quad (57)$$

Running from $E_\pi = \pi \approx 3.14$ to $E_Z \approx 18.38$ with $b_0 = 9$ ($n_f = 3$) from E_π to E_c , then with $b_0 = 25/3$ from E_c to E_b , then $b_0 = 23/3$ from E_b to E_Z :

$$\Delta_{\pi \rightarrow c} = \frac{9 \times 0.79334}{2\pi} (16.136 - 3.142) = 1.1362 \times 12.994 \approx 14.76, \quad (58)$$

$$\Delta_{c \rightarrow b} = 1.0522 \times (17.530 - 16.136) = 1.0522 \times 1.394 \approx 1.47, \quad (59)$$

$$\Delta_{b \rightarrow Z} = 0.9683 \times (18.384 - 17.530) = 0.9683 \times 0.854 \approx 0.83. \quad (60)$$

Total: $\Delta = 14.76 + 1.47 + 0.83 = 17.06$. Predicted $\alpha_s^{-1}(M_Z) = \alpha_s^{-1}(E_\pi) + \Delta = (1/0.2880) + 17.06 = 3.472 + 17.06 = 20.53$, giving $\alpha_s^{\text{pred}}(M_Z) \approx 1/20.53 \approx 0.0487$.

Closer to PDG (0.1179) than Routes A or B, but still off by $\sim 2.4\times$.

Remark 6.6 (Structural motivation for Route C). Route C uses $\text{FRAC}_{\text{bulk}}/\pi = 4\pi^2/\mu_0$ rather than $\text{FRAC}_{\text{bulk}}/(h^\vee(F_4)\pi) = 4\pi^2/(9\mu_0)$ of Addendum 86. The difference is the factor of $h^\vee(F_4) = 9$. Route C sets the coupling at the natural $J_3(\mathbb{O})$ scale $E = \pi$, which is the floor of the spectral coordinate (the electron mass). The $4\pi^2/\mu_0 \approx 0.288$ value is in the strong-coupling regime but not at a Landau pole. The subsequent IR running to M_Z with the SM β -function gives ~ 0.049 , still off by $2.4\times$.

6.4 Summary: none of the routes closes

Table 1: Comparison of route predictions for $\alpha_s(M_Z)$. PDG: 0.1179 ± 0.0009 .

Route	Boundary	E_0	$\alpha_s^{\text{pred}}(M_Z)$	Ratio to PDG
A86 (F_4 LO, GUT)	$9\mu_0/(4\pi^2)$	60.0	0.0147	$0.125\times$
A90 (E_6 consistent)	1/47.14 (by construction)	62.1	0.1179	$1.000\times$
Route A (IR, Λ_{QCD})	$9\mu_0/(4\pi^2)$	10.67	0.025	$0.21\times$
Route C (spectral floor)	$4\pi^2/\mu_0$	π	0.049	$0.41\times$

The E_6 -consistent result in Table 1 is tautological: it was constructed by requiring consistency of the running equations, not derived from the $J_3(\mathbb{O})$ geometry. The open question is whether the TOE provides a first-principles derivation of $\alpha_s^{-1}(E_{\text{GUT}}^{E_6}) = 47.14$ from the F_4 structure alone.

7 Identification of open problems

This addendum refines and extends the open-problem catalogue of Addendum 86.

Definition 7.1 (OP- α_s -1 (revised)). The $F_4 \rightarrow G_2 \rightarrow \text{SU}(3)$ threshold correction to $\alpha_s^{-1}(E_{\text{GUT}})$ required for consistency with the E_6 running system is:

$$\delta\alpha_s^{-1}_{\text{thresh}} = \alpha_s^{-1}(E_{\text{GUT}}^{E_6})_{E_6\text{-consistent}} - \alpha_s^{-1}(E_{\text{GUT}})_{F_4\text{-LO}} \approx 47.14 - 31.24 = 15.90. \quad (61)$$

This replaces the estimate ≈ 6.24 from Addendum 86 Remark 3.3, which was based on comparison with the $\text{SU}(5)$ -inferred boundary. The correct comparison is with the E_6 -self-consistent boundary; the threshold correction is $\sim 2.5\times$ larger than previously estimated.

Definition 7.2 (OP- α_s -2 (revised)). The running-interval problem is now characterised as: the 1-loop $\text{SU}(3)$ running from any TOE-motivated GUT-scale boundary gives $\alpha_s^{-1}(M_Z) \gg 8.48$ unless the boundary coupling is also adjusted to the E_6 -consistent value 47.14. The adjustment $\delta\alpha_s^{-1} = 15.90$ required at the boundary is the content of OP- α_s -1 (revised). OP- α_s -2 is subsumed into OP- α_s -1: once the boundary is correct, the running closes.

Definition 7.3 (OP- α_s -3: TOE hypercharge normalisation in the GUT embedding). The hypercharge coupling in the TOE is $\alpha_Y = \alpha_{\text{EM}}/\cos^2\theta_W$. The GUT-embedding normalisation factor k_1 (which equals $5/3$ in $\text{SU}(5)$) must be derived from the $J_3(\mathbb{O})$ block structure. This requires identifying how the x_{12} Peirce block fraction π/μ_0 relates to the GUT-normalised hypercharge $\text{Tr}[Y^2]$. The TOE Weinberg angle $1/(1+\pi)$ implies $\tan^2\theta_W = 1/\pi$, giving $k_1^{\text{TOE}} = k_1^{\text{SU}(5)} \cdot \pi/3 \approx 5/(3 \cdot \pi) \cdot \pi = 5/3$: coincidentally the same as $\text{SU}(5)$ if k_1^{TOE} is defined as $(5/3)\tan^2\theta_W/(3/5) = 5/(3) \cdot (1/\pi)/(3/5) = 25/(9\pi) \approx 0.884$. This calculation is inconsistent (circular); the correct derivation of k_1^{TOE} is left open.

Definition 7.4 (OP- α_s -4 and OP- α_s -5). OP- α_s -4: Non-perturbative coset-mode contribution from F_4/G_2 at the GUT scale (Route B, Section 6). OP- α_s -5: Spectral-flow placement of the strong coupling boundary at $E = \pi$ or Λ_{QCD} (Route C, Section 6). Both are structurally motivated but neither is derivationally established.

8 Golden-ratio interlocks

Proposition 8.1 ($\sin^2\theta_W^{\text{TOE}}/\sin^2\theta_W^{\text{SU}(5)} \approx 1/\phi$). *The ratio of the TOE Weinberg angle to the $SU(5)$ GUT-scale prediction satisfies:*

$$\frac{\sin^2\theta_W^{(1)}}{\sin^2\theta_W^{\text{SU}(5)}} = \frac{0.23192}{0.375} \approx 0.61845, \quad (62)$$

while the golden-ratio reciprocal is $1/\phi = (\sqrt{5} - 1)/2 \approx 0.61803$. The difference is $|0.61845 - 0.61803| = 0.00042$, i.e. 0.07%.

Proof. Numerical comparison; the NLO Weinberg angle from Addendum 88, Theorem 4.2. $\square$

Remark 8.2 (Is this a structural identity?). The identity $\sin^2\theta_W^{(1)} = 3\phi^{-1}/8$ would read: $3/(8\phi) = 3 \times 0.61803/8 = 0.23176$. vs. $\sin^2\theta_W^{(1)} = 0.23192$. Difference: $0.016/0.23192 \approx 0.07\%$. Exact form: $(1 - 4\lambda^2/5)/(1 + \pi) = 3/(8\phi)$ would require $8\phi(1 - 4\lambda^2/5) = 3(1 + \pi)$, i.e. $8 \times 1.61803 \times 0.96039 = 3 \times 4.14159$, i.e. $12.434 = 12.425$. The two sides differ by 0.07%.

Whether this near-identity has a derivational interpretation in the $E_6/F_4/\phi$ geometry (the golden ratio enters E_6 through the Coxeter element and the E_6 root-system lengths) is an open question. We note it as a pattern without asserting a proof.

9 Summary

Table 2: Status of the $\alpha_s(M_Z)$ derivation after Addendum 90. Entries in italics are from Addendum 86; entries in bold are new in this addendum.

Result	Formula / Source	Value	Status
$\alpha_s^{-1}(E_{\text{GUT}})$ F_4 -LO	$9\mu_0/(4\pi^2)$	31.24	<i>Proved (A86)</i>
$\sin^2\theta_W^{(1)}$ TOE NLO	$(1 - 4\lambda^2/5)/(1 + \pi)$	0.23192	<i>CLOSED (A88)</i>
$\alpha_2^{-1}(M_Z)$ TOE	$\sin^2\theta_W^{(1)}/\alpha_{\text{EM}}(M_Z)$	29.66	Proved
$\sin^2\theta_W^{\text{SU}(5)}$	$3/8$	0.375	Known
TOE $\neq$ SU(5) incompatibility	Cor. 2.2	—	Proved
$E_{\text{GUT}}^{E_6}$ from $\alpha_2 = \alpha_s$	Prop. 3.3	62.12	Proved
α_{GUT}^{-1} E_6 -consistent	Thm. 3.5	47.14	Proved (tautological)
Running gap (multi-thresh.)	Thm. 4.2	$\Gamma \approx 0$	Proved (by construction)
Revised OP- α_s -1 threshold	Def. 7.1	$\delta\alpha_s^{-1} = 15.90$	Open (refined)
$\sin^2\theta_W^{(1)}/\sin^2\theta_W^{\text{SU}(5)} \approx 1/\phi$	Prop. 8.1	0.618	Pattern (not proved)
Route A: IR boundary	Conj. 6.1	$\alpha_s(M_Z) \approx 0.025$	Open
Route C: spectral floor	Conj. 6.5	$\alpha_s(M_Z) \approx 0.049$	Open
$\alpha_s(M_Z)$ from TOE first principles	—	—	NOT CLOSED
PDG $\alpha_s(M_Z)$	PDG 2024	0.1179 ± 0.0009	Reference

The principal results are:

- **E_6 incompatibility with SU(5) (Corollary 2.2):** The TOE Weinberg angle $\sin^2\theta_W^{(1)} \neq 3/8$ implies that the $J_3(\mathbb{O})$ electroweak sector does not embed into $E_6 \supset \text{SO}(10) \supset \text{SU}(5)$ in the standard way. The hypercharge normalisation $k_1^{\text{TOE}} \neq 5/3$.
- **E_6 -consistent boundary (Theorem 3.5):** Requiring $\alpha_2 = \alpha_s$ at some E_{GUT} using the TOE Weinberg angle as input gives $E_{\text{GUT}}^{E_6} \approx 62.12$ and $\alpha_{\text{GUT}}^{-1} \approx 47.14$. This is a tautological consistency check, not a first-principles derivation.

- **Revised OP- α_s -1 (Definition 7.1):** The $F_4 \rightarrow G_2 \rightarrow \text{SU}(3)$ threshold correction required to match the E_6 GUT boundary is $\delta\alpha_s^{-1} \approx 15.90$, nearly $2.5\times$ larger than the estimate ≈ 6.24 in Addendum 86.
- **Resolution candidates (Section 6):** Route A (IR boundary at Λ_{QCD}) gives $\alpha_s(M_Z) \approx 0.025$; Route C (spectral floor at $E = \pi$) gives $\alpha_s(M_Z) \approx 0.049$. Neither closes the gap. Route B (non-perturbative coset modes) is not viable at leading order.
- **Status: NOT CLOSED.** No first-principles TOE derivation of $\alpha_s(M_Z)$ is achieved. The F_4 boundary condition from Addendum 86 and the E_6 -consistency condition from this addendum are incompatible unless the F_4 -LO estimate is corrected by $\delta\alpha_s^{-1} \approx 15.90$ from the threshold chain.

References

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