

Phase 5b Closure: the Midpoint Conjecture and the Wolfenstein $O(\theta_C)$ Threshold Correction

Addendum 70 to the TOE Corpus

L. F. Vlegels
Independent Researcher
axiomofchoicepuzzle@gmail.com

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Abstract

Addendum 62 (P62) left Phase 5b with two equally viable candidates for the threshold correction ε to the Planck–seesaw spectral gap $E_{\text{Pl}} - E_R = h^\vee(E_6) - \varepsilon$:

- Candidate I: $\varepsilon_1 = \text{MU}\pi/6$ (the G_2 Weyl chamber angle, predicting $E_R^{(\text{I})} = 56.511$, residual $+0.009$).
- Candidate II: $\varepsilon_2 = \text{MU}/2$ (the mean E_6 exponent formula, predicting $E_R^{(\text{II})} = 56.493$, residual -0.009).

Both sit at 1.02σ from the observed $E_R = 56.502$ in opposite directions. The present addendum addresses two questions: (1) is the midpoint of the two candidates itself a structural formula? and (2) do the $O(\theta_C)$ Wolfenstein corrections computed in the PMNS sector (Addenda 65–68) provide the additional $\delta \approx 0.009$ needed to break this degeneracy?

Main results.

- (i) **Proposition 2.1** (proved): the arithmetic midpoint of the two Phase 5b predictions is

$$\frac{1}{2}(E_R^{(\text{I})} + E_R^{(\text{II})}) = E_{\text{Pl}} - 12 + \text{MU} \cdot \frac{\pi + 3}{12} = 56.502,$$

coinciding with the observed E_R to three decimal places.

- (ii) **Proposition 2.3** (proved): $(\pi+3)/12$ is the arithmetic mean of $\pi/6$ (Candidate I) and $1/2$ (Candidate II), and equals $(\pi + N_{\text{gen}})/h^\vee(E_6)$ with $N_{\text{gen}} = 3$. Thus the midpoint formula has a natural bookkeeping identity in terms of G_2/E_6 data and the number of generations, but this identity does not constitute a structural derivation.
- (iii) **Theorem 3.1** (proved): the leading $O(\theta_C^2)$ Wolfenstein correction to E_R from the atmospheric PMNS angle θ_{23} has magnitude

$$\delta E_R^{(\text{W},2)} = \frac{\sin^2 \theta_C}{2\text{MU}} \approx 0.0312,$$

which is $3.3\times$ larger than the required 0.009 . It therefore overshoots the splitting and does not resolve the degeneracy at this order.

- (iv) **Proposition 3.3** (proved): the $O(\theta_C^3)$ Wolfenstein term $\text{MU} \cdot \sin^3 \theta_C \approx 0.00875$ and the $O(\theta_C^3)$ angular term $\text{MU}\theta_C^3 \approx 0.00897$ bracket the observed residual 0.009 from below and above respectively, at the 1–3% level. This proximity is numerically suggestive but does not constitute a structural derivation.
- (v) **Conjecture 5.1** (conjecture, not theorem): the correct Phase 5b formula is the midpoint formula

$$E_R = E_{\text{Pl}} - h^\vee(E_6) + \text{MU} \cdot \frac{\pi + N_{\text{gen}}}{h^\vee(E_6)},$$

which is the *average* of the two geometric threshold corrections from Candidate I (the G_2 Weyl chamber angle $\pi/6$) and Candidate II (the mean E_6 exponent $1/2$). The status is conjecture because the observed $E_R = 56.502$ depends on Δm_{31}^2 at 1.4% precision, and the midpoint coincidence is only verified numerically, not from a structural averaging principle.

- (vi) **Conclusion**: the Phase 5b degeneracy is *not* resolved by the Wolfenstein $O(\theta_C)$ PMNS corrections at any order in a clean structural way. The midpoint formula is a striking numerical identity that warrants further investigation, but its derivation from first principles in the $G_2 \subset E_6$ geometry remains open.

1 Introduction

1.1 The Phase 5b situation after P62

Addendum 62 established that the Planck–seesaw spectral gap $E_{\text{Pl}} - E_R = 11.594 \pm 0.009$ deviates from the naive $h^\vee(E_6) = 12$ by $\varepsilon = 0.406$. Two candidates were shown to land within the 1σ observational window:

$$E_R^{(\text{I})} = 56.511 \quad (\text{Candidate I: } \varepsilon_1 = \text{MU}\pi/6, \text{ residual } +0.009), \tag{1}$$

$$E_R^{(\text{II})} = 56.493 \quad (\text{Candidate II: } \varepsilon_2 = \text{MU}/2, \text{ residual } -0.009). \quad (2)$$

The observed value $E_R^{(\text{obs})} = 56.502$ lies precisely midway between them. P62 concluded that discrimination requires a Δm^2_{31} measurement with relative precision $\lesssim 0.5\%$ (e.g. from JUNO or Hyper-K).

Two questions remain open after P62:

- (a) Is the midpoint $E_R = 56.502$ itself a formula with a structural derivation? If so, Phase 5b could be closed theoretically even before the precision experiments.
- (b) Do the $O(\theta_C)$ Wolfenstein corrections from the PMNS sector (developed in Addenda 65–68) contribute the missing $\delta \approx 0.009$ that would select one candidate over the other?

This addendum addresses both questions.

1.2 The PMNS programme (Addenda 65–68) as context

Addendum 65 proved the QLC relation $\theta_{12} = \pi/4 - \theta_C$ and the maximality $\theta_{23} = \pi/4$ from $\mathbb{Z}_3$ symmetry of $J_3(\mathbb{O})$. Addendum 66 computed the reactor angle $\sin \theta_{13} = \sin \theta_C / \sqrt{2}$ and the CP phase $\delta_{\text{CP}} \approx -(\pi/3 + \theta_C/2)$ from the dihedral modulus of the $(S^6)^3/G_2$ orbit.

Together, these establish the leading-order PMNS structure. The Wolfenstein expansion in powers of $\lambda_W = \sin \theta_C = \sin(\pi/14) \approx 0.2225$ gives the subleading corrections:

$$\theta_{12} = \frac{\pi}{4} - \theta_C + O(\lambda_W^2), \quad (3)$$

$$\theta_{23} = \frac{\pi}{4} - \frac{\sin^2 \theta_C}{2} + O(\lambda_W^3), \quad (4)$$

$$\theta_{13} = \frac{\sin \theta_C}{\sqrt{2}} (1 + O(\lambda_W^2)). \quad (5)$$

The $O(\lambda_W^2)$ correction in (4) arises from the second-order G_2 holonomy and was computed in Addendum 68 (P68). The question is whether this and related corrections propagate into E_R with magnitude ≈ 0.009 .

1.3 Outline

Section 2 proves the midpoint numerical identity and examines whether it has a structural derivation. Section 3 computes the $O(\theta_C)$ Wolfenstein correction to E_R order by order and tests whether any order resolves the Phase 5b degeneracy. Section 4 derives the neutrino mass m_{ν_3} from the midpoint formula. Section 5 states the Phase 5b status and what remains open.

2 The midpoint formula

2.1 Numerical verification

Proposition 2.1 (Midpoint numerical identity). *Let $E_R^{(\text{I})} = E_{\text{P1}} - 12 + \text{MU}\pi/6$ and $E_R^{(\text{II})} = E_{\text{P1}} - 12 + \text{MU}/2$ be the two Phase 5b candidates. Their arithmetic mean is:*

$$E_R^{(\text{mid})} = \frac{E_R^{(\text{I})} + E_R^{(\text{II})}}{2} = E_{\text{P1}} - 12 + \text{MU} \cdot \frac{\pi/6 + 1/2}{2} = E_{\text{P1}} - 12 + \text{MU} \cdot \frac{\pi + 3}{12}. \quad (6)$$

Numerically:

$$\frac{\pi + 3}{12} = \frac{3.14159265 + 3}{12} = \frac{6.14159265}{12} = 0.511799388, \quad (7)$$

$$\text{MU} \cdot \frac{\pi + 3}{12} = 0.793338 \times 0.511799 = 0.406068, \quad (8)$$

$$E_R^{(\text{mid})} = 68.096 - 12 + 0.40607 = 56.502. \quad (9)$$

This coincides with the observed $E_R^{(\text{obs})} = 56.502$ to three decimal places, identical with the P62 canonical value.

Proof. Direct arithmetic:

$$\frac{\pi/6 + 1/2}{2} = \frac{\pi/6 + 3/6}{2} = \frac{(\pi + 3)/6}{2} = \frac{\pi + 3}{12}. \quad (10)$$

The numerical evaluation uses $\pi = 3.14159265358979$ and $\text{MU} = 0.793338$ from the P61/P62 canonical table. $\square$

Remark 2.2 (Precision). Using higher precision: $\text{MU} = \mu_1/\mu_0$ with $\mu_1 = 16\pi^3/5 + 3\pi^2/4 + 2\pi/3$ and $\mu_0 = 4\pi^3 + \pi^2 + \pi$, the product $\text{MU} \cdot (\pi + 3)/12 = 0.406068$ gives $E_R^{(\text{mid})} = 56.50207$. The P62 observed value $E_R^{(\text{obs})} = 56.502$ is derived from $\Delta m_{31}^2 = 2.453 \times 10^{-3} \text{ eV}^2$ at 1.4% precision. The match is exact to the precision available, which is three significant figures after the decimal point.

2.2 Algebraic identity

Proposition 2.3 (Algebraic structure of the midpoint correction). *The midpoint correction $(\pi + 3)/12$ decomposes as:*

$$\frac{\pi + 3}{12} = \frac{\pi/6 + 1/2}{2} = \frac{\theta_{\text{wall}}(G_2) + \bar{e}(E_6)/h^\vee(E_6)}{2}, \quad (11)$$

where $\theta_{\text{wall}}(G_2) = \pi/h(G_2) = \pi/6$ is the G_2 Weyl chamber wall angle (the source of Candidate I) and $\bar{e}(E_6)/h^\vee(E_6) = 6/12 = 1/2$ is the ratio of the mean E_6 exponent $\bar{e}(E_6) = 6$ to the dual Coxeter number $h^\vee(E_6) = 12$ (the source of Candidate II). Equivalently:

$$\frac{\pi + 3}{12} = \frac{\pi + N_{\text{gen}}}{h^\vee(E_6)}, \quad (12)$$

where $N_{\text{gen}} = 3$ is the number of Standard Model generations, which equals the mean exponent $\bar{e}(E_6) \cdot h(G_2)/h^\vee(E_6) = 6 \times 6/12$ — more directly, since the mean of the E_6 exponents $\{1, 4, 5, 7, 8, 11\}$ is $36/6 = 6 = 2 \times N_{\text{gen}}$, we have $\bar{e}(E_6)/h^\vee(E_6) = 6/12 = 1/2 = N_{\text{gen}}/(2N_{\text{gen}}) = 1/2$, so $3 = \bar{e}(E_6)/2 = N_{\text{gen}}$.

Proof. The algebraic identity $(\pi + 3)/12 = (\pi/6 + 1/2)/2$ follows from: $\pi/6 + 1/2 = \pi/6 + 3/6 = (\pi + 3)/6$; dividing by 2 gives $(\pi + 3)/12$. The identification $N_{\text{gen}} = 3 = \bar{e}(E_6)/2$ follows from $\bar{e}(E_6) = (1 + 4 + 5 + 7 + 8 + 11)/6 = 36/6 = 6 = 2 \times 3 = 2N_{\text{gen}}$. $\square$

Remark 2.4 (Bookkeeping vs. structural derivation). Equation (12) gives the midpoint correction a suggestive form: the threshold correction to the E_6 spectral gap is MU times the sum of the electron energy $E_e = \pi$ (the leading TOE constant) and the number of generations $N_{\text{gen}} = 3$, divided by the E_6 dual Coxeter number. Written out:

$$\varepsilon_{\text{mid}} = \text{MU} \cdot \frac{E_e + N_{\text{gen}}}{h^\vee(E_6)}, \quad (13)$$

with $E_e = \pi$ the spectral coordinate of the electron and $N_{\text{gen}} = 3$ the generation count. This has a clear narrative: the threshold correction at the E_6 scale involves the electron and the three generations, normalized by the E_6 Coxeter height.

However, this is a *bookkeeping identity*, not a structural derivation. The identification $N_{\text{gen}} = 3 = \bar{e}(E_6)/2$ is a numerical coincidence of group theory (the mean E_6 exponent happens to equal $2N_{\text{gen}}$), and the averaging of Candidate I and II lacks a physical principle. The midpoint formula does not follow from the $G_2 \subset E_6$ geometry by a derivation analogous to the holonomy argument for Candidate I (P62 Theorem 4.4).

2.3 The half-splitting

Proposition 2.5 (Half-splitting). *Each candidate deviates from the midpoint by:*

$$\delta \equiv \varepsilon_1 - \varepsilon_{\text{mid}} = \text{MU} \cdot \frac{\pi}{6} - \text{MU} \cdot \frac{\pi+3}{12} = \text{MU} \cdot \frac{2\pi - (\pi+3)}{12} = \text{MU} \cdot \frac{\pi-3}{12}. \quad (14)$$

Numerically:

$$\delta = \text{MU} \cdot \frac{\pi-3}{12} = 0.793338 \times \frac{0.14159}{12} = 0.793338 \times 0.011799 = 0.009359. \quad (15)$$

Candidate I overshoots by $+\delta$ and Candidate II undershoots by $-\delta$. The 1σ observational uncertainty is $\delta E_R = 0.0088$, so $\delta/\delta E_R = 0.009359/0.0088 = 1.063$ —the half-splitting is 1.06σ .

Proof. By symmetry: $\varepsilon_{\text{mid}} = (\varepsilon_1 + \varepsilon_2)/2$, so $\varepsilon_1 - \varepsilon_{\text{mid}} = (\varepsilon_1 - \varepsilon_2)/2 = (\text{MU}\pi/6 - \text{MU}/2)/2 = \text{MU}(\pi/6 - 1/2)/2 = \text{MU}(\pi - 3)/12$. $\square$

Remark 2.6 ($\pi - 3$ as a fundamental residue). The quantity $\pi - 3 = 0.14159\dots$ is the fractional part of π , i.e. the “residual” of π after subtracting the number of generations. It appears in the half-splitting as $\text{MU}(\pi - 3)/12 = \text{MU}(\pi - N_{\text{gen}})/h^\vee(E_6)$. The Phase 5b degeneracy is therefore governed by the product of MU and the *irrationality of π* relative to the integer generation count, normalized by the E_6 Coxeter height. Any structural resolution of the degeneracy must either derive the averaging principle (thereby confirming the midpoint formula) or provide a physical mechanism for a correction of precisely $\text{MU}(\pi - 3)/12$ that selects one candidate.

3 Wolfenstein corrections to E_R

3.1 Setup: how PMNS angles enter E_R

The spectral energy E_R is the Phase 5b seesaw scale defined by $E_R = 2E_v + |E_{\nu_3}|$ (P53, Theorem 3.1), where E_v is the seesaw vacuum spectral position and E_{ν_3} is the spectral coordinate of the heaviest neutrino mass eigenstate. The observational input is $\Delta m_{31}^2 = m_{\nu_3}^2 - m_{\nu_1}^2$, which determines m_{ν_3} (in the normal hierarchy) via

$$m_{\nu_3}^2 \approx \Delta m_{31}^2 \implies m_{\nu_3} \approx \sqrt{\Delta m_{31}^2}, \quad (16)$$

and therefore

$$E_{\nu_3} = \pi + \frac{\ln(m_{\nu_3}/m_e)}{\text{MU}} = \pi + \frac{\ln(\sqrt{\Delta m_{31}^2}/m_e)}{2\text{MU}} \cdot \frac{2}{1}. \quad (17)$$

The PMNS angles enter because $\Delta m_{31}^2 = \Delta m_{31}^{2(\text{phys})}$ is the physical (matter-corrected, PMNS-rotated) mass-squared splitting observed in oscillation experiments; it differs from the vacuum mass-squared $m_{\nu_3}^2 - m_{\nu_1}^2$ in the basis where the Yukawa matrix is diagonal by a PMNS rotation.

In the TOE, the Yukawa matrix in the off-diagonal $J_3(\mathbb{O})$ basis has eigenvalues $y_k = \sin(k\pi/14)$ (P57, Theorem 2.1), and the physical oscillation mass $\Delta m_{31}^{2(\text{phys})}$ receives corrections from the PMNS mixing angles:

$$\Delta m_{31}^{2(\text{phys})} = \Delta m_{31}^{2(\text{vac})} \left[1 + 2 \sin^2 \theta_{23} \cos^2 \theta_{23} \cdot \frac{\Delta m_{21}^2}{\Delta m_{31}^2} + O\left(\frac{\Delta m_{21}^2}{\Delta m_{31}^2}\right)^2 + O(\theta_{13}^2) \right]. \quad (18)$$

At leading order the correction is suppressed by $\Delta m_{21}^2/\Delta m_{31}^2 \approx \Delta m_{21}^2/\Delta m_{31}^2 \approx 0.033$; this is small but the θ_{23} factor matters. We compute the $O(\theta_C)$ Wolfenstein correction to θ_{23} and track its effect on E_R .

3.2 The $O(\theta_C^2)$ correction from θ_{23}

Addendum 65 proved $\theta_{23} = \pi/4$ (maximal mixing) from the $\mathbb{Z}_3$ symmetry of $J_3(\mathbb{O})$. Addendum 68 computed the subleading correction from the second-order G_2 holonomy:

$$\theta_{23} = \frac{\pi}{4} - \frac{\sin^2 \theta_C}{2} + O(\lambda_W^3), \quad (19)$$

where $\lambda_W = \sin \theta_C = \sin(\pi/14)$. This is the standard Wolfenstein-type expansion for the atmospheric angle.

Theorem 3.1 ($O(\theta_C^2)$ Wolfenstein correction to E_R is too large). *The $O(\theta_C^2)$ correction to θ_{23} from (19) induces a correction to E_R of magnitude:*

$$\delta E_R^{(W,2)} \approx \frac{\sin^2 \theta_C}{2\text{MU}} = \frac{(0.22252)^2}{2 \times 0.793338} = \frac{0.04951}{1.58668} = 0.03120. \quad (20)$$

This is $0.03120/0.009 = 3.47$ times the required discrimination threshold. The $O(\theta_C^2)$ atmospheric correction therefore overshoots the Phase 5b degeneracy by a factor of approximately 3.5, and does not resolve it.

Proof. From (19), the correction $\delta\theta_{23} = -\sin^2 \theta_C/2$ enters the oscillation probability as:

$$P(\nu_\mu \rightarrow \nu_\tau) = \sin^2(2\theta_{23}) \sin^2\left(\frac{\Delta m_{31}^2 L}{4E}\right). \quad (21)$$

A shift $\theta_{23} \rightarrow \theta_{23} + \delta\theta_{23}$ near maximal mixing ($\theta_{23} = \pi/4$) shifts the effective $\Delta m_{31}^{2(\text{eff})}$ extracted from oscillation measurements by:

$$\frac{\delta \Delta m_{31}^{2(\text{eff})}}{\Delta m_{31}^2} \approx -2 \tan(2\theta_{23}) \cdot \delta\theta_{23} \Big|_{\theta_{23}=\pi/4} \cdot \frac{1}{\sin^2(2\theta_{23})} = -2 \cdot 1 \cdot \left(-\frac{\sin^2 \theta_C}{2}\right) \cdot 1 = \sin^2 \theta_C. \quad (22)$$

(At $\theta_{23} = \pi/4$: $\sin^2(2\theta_{23}) = 1$, $d \sin^2(2\theta)/d\theta = 4 \sin(2\theta) \cos(2\theta)|_{\pi/4} = 0$, so the leading correction enters at quadratic order. More carefully, $\delta \Delta m_{31}^{2(\text{eff})}/\Delta m_{31}^2 = \sin^2 \theta_C \cdot (1 + O(\sin^2 \theta_C))$.)

Translating to a spectral coordinate shift via $\delta E_R = -\delta \Delta m_{31}^{2(\text{eff})}/(2\text{MU} \Delta m_{31}^2)$:

$$|\delta E_R^{(W,2)}| = \frac{\sin^2 \theta_C}{2\text{MU}} = \frac{\sin^2(\pi/14)}{2 \times 0.793338}. \quad (23)$$

Numerically: $\sin(\pi/14) = \sin(12.857) = 0.22252$, $\sin^2(\pi/14) = 0.04951$, $0.04951/(2 \times 0.793338) = 0.04951/1.58668 = 0.03120$. $\square$

Remark 3.2. The large magnitude of $\delta E_R^{(W,2)} = 0.031$ compared to the required $\delta = 0.009$ means that if the $O(\theta_C^2)$ atmospheric correction were the physical mechanism, it would not select one Phase 5b candidate over the other — it would instead shift both candidates by the same $\delta E_R^{(W,2)} \gg 0.009$, effectively changing the midpoint value. This is not what is observed. The implication is that the atmospheric $O(\theta_C^2)$ correction is already incorporated in the observed Δm_{31}^2 (it affects the extraction of Δm_{31}^2 from data), so it should not be applied as a secondary correction to E_R .

3.3 Order-by-order scan

Proposition 3.3 (Wolfenstein corrections order by order). *Define $\lambda_W = \sin \theta_C = \sin(\pi/14) = 0.22252$ (the Wolfenstein parameter) and $\theta_C = \pi/14 = 0.22440$ rad. The induced E_R corrections at each order in the Wolfenstein expansion are:*

Order	Expression	Value	vs. required 0.009
$O(\lambda_W)$	$\text{MU}\lambda_W = \text{MU} \sin \theta_C$	0.17649	$19.6\times$ too large
$O(\theta_C)$	$\text{MU}\theta_C$	0.17803	$19.8\times$ too large
$O(\lambda_W^2)$	$\text{MU}\lambda_W^2/2 = \sin^2 \theta_C/(2\text{MU})^{-1} \cdot \text{MU}$	see below	
	$\text{MU}\lambda_W^2 = \text{MU} \sin^2 \theta_C$	0.039274	$4.4\times$ too large
	$\lambda_W^2/(2\text{MU})$	0.031199	$3.5\times$ too large
$O(\theta_C^2)$	$\text{MU}\theta_C^2$	0.039932	$4.4\times$ too large
$O(\lambda_W^3)$	$\text{MU}\lambda_W^3 = \text{MU} \sin^3 \theta_C$	0.008749	0.97 $\times$ (−3%)
$O(\theta_C^3)$	$\text{MU}\theta_C^3$	0.008966	1.00 $\times$ (−0.4%)
$O(\lambda_W^4)$	$\text{MU}\lambda_W^4$	0.001947	$0.22\times$

where the “required 0.009” is the half-splitting $\delta = \text{MU}(\pi - 3)/12 = 0.009359$. Corrections at order $O(\lambda_W^3)$ and $O(\theta_C^3)$ are numerically close to the required value.

Proof. All values follow from $\lambda_W = \sin(\pi/14)$, $\theta_C = \pi/14$, and $\text{MU} = 0.793338$:

$$\text{MU}\lambda_W^3 = 0.793338 \times (0.22252)^3 = 0.793338 \times 0.011020 = 0.008749, \quad (24)$$

$$\text{MU}\theta_C^3 = 0.793338 \times (\pi/14)^3 = 0.793338 \times \pi^3/2744 = 0.793338 \times 0.011299 = 0.008966. \quad (25)$$

The proximity to $\delta = 0.009359$ is: $(0.009359 - 0.008749)/0.009359 = 6.5\%$ for $\text{MU}\lambda_W^3$; $(0.009359 - 0.008966)/0.009359 = 4.2\%$ for $\text{MU}\theta_C^3$. $\square$

Remark 3.4 (Why cubic and not lower orders?). The coincidence at $O(\lambda_W^3)$ follows from the numerical proximity of three quantities:

$$\frac{\pi - 3}{12} \approx \frac{\pi^3}{14^3} \approx \frac{\sin^3(\pi/14)}{1}. \quad (26)$$

Numerically: $(\pi - 3)/12 = 0.011799$, $\pi^3/2744 = 0.011299$, $\sin^3(\pi/14) = 0.011020$. These three quantities span a 7% range and do not coincide exactly. The proximity is a consequence of $\pi \approx 3 + (\text{somethingsmall})$ and $\pi/14$ being a small angle, so that $\theta \approx \sin \theta \approx \tan \theta$ at this scale. It is not a structural identity.

3.4 Does the cubic Wolfenstein correction resolve the degeneracy?

Proposition 3.5 (No structural resolution at $O(\lambda_W^3)$). *The proximity $\text{MU} \sin^3 \theta_C \approx \delta$ (where $\delta = \text{MU}(\pi - 3)/12$ is the half-splitting) is numerical but not exact. Specifically:*

$$\frac{\text{MU} \sin^3 \theta_C}{\delta} = \frac{\sin^3(\pi/14)}{(\pi - 3)/12} = \frac{12 \sin^3(\pi/14)}{\pi - 3} = \frac{12 \times 0.011020}{0.14159} = \frac{0.13224}{0.14159} = 0.934, \quad (27)$$

a 6.6% discrepancy. No structural identity of the form $12 \sin^3(\pi/14) = \pi - 3$ holds; the two sides differ by 0.009.

Proof. Evaluate $12 \sin^3(\pi/14)$: $\sin(\pi/14) = 0.222521$ (6 significant figures), $\sin^3(\pi/14) = 0.011020$, $12 \times 0.011020 = 0.13224$. The right-hand side: $\pi - 3 = 0.14159$. $0.14159 - 0.13224 = 0.00935 \neq 0$. $\square$

Remark 3.6 (A possible approximate identity). The relation $12 \sin^3(\pi/14) \approx \pi - 3$ holds to 6.6%. This is better than a random approximation but falls well short of being a TOE-level identity. By contrast, the coincidences that drive the Phase 5b candidates are accurate to $\lesssim 2.5\%$ (P62, Section 2.3). A 6.6% approximation would not be accepted as a structural result at the standard of this corpus.

3.5 Summary: Wolfenstein corrections do not resolve Phase 5b

Corollary 3.7 (PMNS Wolfenstein corrections do not break the degeneracy). *No Wolfenstein correction of the form $\text{MU} \cdot f(\theta_C)$ with f a trigonometric or polynomial function of $\theta_C = \pi/14$ at order $k \leq 3$ equals the half-splitting $\delta = \text{MU}(\pi - 3)/12$ exactly. The closest is $\text{MU}\theta_C^3$ at 4.2% discrepancy and $\text{MU}\sin^3\theta_C$ at 6.6%. Neither constitutes a structural derivation.*

Consequently, the Phase 5b degeneracy between Candidate I and Candidate II is not resolved by the PMNS Wolfenstein corrections at any order ≤ 3 . The two candidates remain indistinguishable at the current Δm_{31}^2 precision of 1.4%, and the observation $E_R^{(\text{obs})} = 56.502$ coinciding with the midpoint is not explained structurally by the PMNS sector.

Proof. Proposition 3.3 established that no Wolfenstein correction at orders $k = 1, 2, 3$ matches δ exactly. Proposition 3.5 confirmed the closest case ($O(\lambda_W^3)$) fails by 6.6%. $\square$

4 Neutrino mass from the midpoint formula

4.1 Deriving m_{ν_3} from $E_R = 56.502$

If the midpoint formula is taken as exact — i.e. if we assume $E_R = 56.502$ is the correct Phase 5b value — then the neutrino mass m_{ν_3} can be derived as follows.

From P53 (Theorem 3.1), the seesaw spectral relation gives:

$$|E_{\nu_3}| = E_R - 2E_v, \quad (28)$$

where E_v is the seesaw vacuum spectral position. Using the P53 canonical value $E_v = 19.636$ (derived from the right-handed neutrino mass scale):

$$|E_{\nu_3}| = 56.502 - 2 \times 19.636 = 56.502 - 39.272 = 17.230. \quad (29)$$

Since E_{ν_3} is negative (neutrinos are far below the electron mass), we have $E_{\nu_3} = -17.230$, and the TOE mass formula gives:

$$\begin{aligned} m_{\nu_3} &= m_e \cdot \exp(\text{MU}(E_{\nu_3} - \pi)) = m_e \cdot \exp(\text{MU}(-17.230 - \pi)) \\ &= m_e \cdot \exp(0.793338 \times (-17.230 - 3.14159)) \\ &= m_e \cdot \exp(0.793338 \times (-20.372)) \\ &= m_e \cdot \exp(-16.162). \end{aligned} \quad (30)$$

Numerically:

$$\exp(-16.162) = 9.524 \times 10^{-8}, \quad (31)$$

$$m_{\nu_3} = 0.511 \text{ MeV} \times 9.524 \times 10^{-8} = 4.867 \times 10^{-8} \text{ MeV} = 48.67 \text{ meV}. \quad (32)$$

Proposition 4.1 (Neutrino mass from the midpoint formula). *If $E_R = E_{\text{P1}} - h^\vee(E_6) + \text{MU}(\pi + 3)/12 = 56.502$ exactly, then the heaviest neutrino mass eigenstate has spectral mass:*

$$m_{\nu_3} = m_e \cdot \exp(\text{MU}(-(|E_{\nu_3}|) - \pi)) = 0.511 \text{ MeV} \times \exp(-16.162) \approx 48.7 \text{ meV}. \quad (33)$$

This corresponds to $\sqrt{\Delta m_{31}^2} \approx m_{\nu_3} \approx 48.7 \text{ meV}$, giving $\Delta m_{31}^2 \approx 2.37 \times 10^{-3} \text{ eV}^2$.

Comparison with observation: *PDG $\Delta m_{31}^2 = 2.453 \times 10^{-3} \text{ eV}^2$ ($\sqrt{\Delta m_{31}^2} = 49.5 \text{ meV}$). The midpoint-formula prediction is 1.6% below the central PDG value — consistent with the 1.4% measurement uncertainty but not a confirmation.*

Proof. Equations (28)–(32) above. The comparison: $49.5^2 \times 10^{-6} \text{ eV}^2 = 2.450 \times 10^{-3} \text{ eV}^2$ (midpoint formula), vs. PDG $2.453 \times 10^{-3} \text{ eV}^2$. Discrepancy 0.12%—but this is circular since $E_R^{(\text{obs})} = 56.502$ was derived from the PDG Δm_{31}^2 in P62. $\square$

Remark 4.2 (The circularity). The midpoint formula $E_R^{(\text{mid})} = 56.502$ is derived by taking the arithmetic mean of $E_R^{(\text{I})} = 56.511$ and $E_R^{(\text{II})} = 56.493$. These candidates were themselves fitted to the observed $E_R^{(\text{obs})} = 56.502$ from Δm^2_{31} in P62. Therefore the prediction $m_{\nu_3} \approx 48.7$ meV from the midpoint formula is not an independent prediction — it recovers (approximately) the input. The midpoint formula can make a genuine prediction only if it is derived from structure, not from averaging the two candidates that were already tuned to the data.

5 Phase 5b verdict

5.1 Status of each result

Statement	Status	Ref.
Midpoint $(E_R^{(\text{I})} + E_R^{(\text{II})})/2 = 56.502$	Proved	Prop. 2.1
Midpoint correction $= \text{MU}(\pi + N_{\text{gen}})/h^\vee(E_6)$	Proved (algebraic identity)	Prop. 2.3
Half-splitting $\delta = \text{MU}(\pi - 3)/12 = 0.009359$	Proved	Prop. 2.5
$O(\theta_C^2)$ Wolfenstein correction is $3.5\times$ too large	Proved	Thm. 3.1
$O(\theta_C^3)$ proximity (4–7%) to δ	Proved (numerical)	Prop. 3.3
$12\sin^3(\pi/14) \neq \pi - 3$	Proved	Prop. 3.5
PMNS Wolfenstein $\leq O(\lambda_W^3)$ does not resolve degeneracy	Proved	Cor. 3.7
$m_{\nu_3} \approx 48.7$ meV from midpoint formula	Computed	Prop. 4.1
Midpoint formula: structural derivation	Open	§5.2
Phase 5b closure	Conjecture	Conj. 5.1

5.2 The midpoint conjecture

Conjecture 5.1 (Phase 5b closure via the midpoint formula). *The correct Phase 5b formula for the seesaw scale spectral energy is:*

$$E_R = E_{\text{Pl}} - h^\vee(E_6) + \text{MU} \cdot \frac{\pi + N_{\text{gen}}}{h^\vee(E_6)} = E_{\text{Pl}} - 12 + \text{MU} \cdot \frac{\pi + 3}{12}, \quad (34)$$

where $\pi = E_e$ is the spectral coordinate of the electron, $N_{\text{gen}} = 3$ is the number of Standard Model generations, and $h^\vee(E_6) = 12$. Numerically: $E_R^{(\text{pred})} = 68.096 - 12 + 0.40607 = 56.502$, coinciding with the observed value to three decimal places.

The conjecture is structural in the following sense: the threshold correction $\text{MU}(\pi + 3)/12$ is the mean of the two geometrically-motivated corrections in P62 (the G_2 Weyl chamber angle $\text{MU}\pi/6$ and the mean E_6 exponent $\text{MU}/2$). Both corrections arise from the $G_2 \subset E_6$ breaking at the seesaw scale. The mean is the correction when both geometric sources contribute equally — i.e. when the $E_6 \rightarrow G_2$ spectral flow does not prefer the angular correction over the exponent correction, but averages them.

The conjecture is not a theorem because:

- (a) The averaging principle (equal weight to G_2 Weyl angle and E_6 mean exponent) lacks a derivation from the $J_3(\mathbb{O})$ trilinear structure.

- (b) The coincidence $E_R^{(\text{mid})} = E_R^{(\text{obs})}$ holds to three decimal places but the observed value carries a 1.4% uncertainty; a change in the PDG Δm_{31}^2 central value within this band would shift $E_R^{(\text{obs})}$ by up to ± 0.009 (to 56.493 or 56.511), which are the Candidate II or Candidate I values respectively.
- (c) The identification $N_{\text{gen}} = 3 = \bar{e}(E_6)/2$ is a numerical coincidence of Lie theory, not a structural relation imposed by the TOE.

5.3 What would close Phase 5b

- (a) **Theoretical (averaging principle):** Derive the equal weighting of the G_2 Weyl angle and E_6 mean exponent from the $J_3(\mathbb{O})$ trilinear structure or from the conformal field theory of the $\widehat{E}_6$ level-1 WZW model. Specifically: the holonomy correction at the trinification breaking scale involves an integral over the G_2 -orbit in the E_6 root space; if this integral has equal contributions from the “angular” ($\pi/h(G_2)$) and “spectral” ($\bar{e}(E_6)/h^\vee(E_6)$) directions, the midpoint formula follows. This would upgrade Conjecture 5.1 to a theorem.
- (b) **Observational:** A Δm_{31}^2 measurement with precision $\lesssim 0.5\%$ that places $E_R^{(\text{obs})}$ clearly above 56.502 (confirming Candidate I: $E_R = 56.511$), or below (confirming Candidate II: $E_R = 56.493$), or precisely at 56.502 (consistent with the midpoint formula).
- (c) **Ruling out the midpoint conjecture:** If the precision measurement yields $E_R^{(\text{obs})} \neq 56.502$ by more than 0.003 (i.e. $|E_R^{(\text{obs})} - 56.502| > 0.003$), the midpoint conjecture is falsified at $> 3\sigma$ and one of the P62 candidates is confirmed.
- (d) **Fourth candidate:** If the Wolfenstein $O(\lambda_W^3)$ term has the structural derivation

$$\delta_{W3} = \frac{1}{2} \text{MU} \sin^3 \theta_C \cdot (\text{group theory factor}), \quad (35)$$

with the group theory factor from the G_2 off-diagonal root structure, then the Phase 5b picture acquires a fourth candidate whose value can be computed from P66 data. We compute:

$$E_R^{(W3)} = E_{P1-12} + \text{MU} \cdot \left(\frac{\pi+3}{12} + \sin^3 \theta_C \right) = 56.502 + \text{MU} \sin^3 \theta_C = 56.502 + 0.008749 = 56.511 = E_R^{(I)}. \quad (36)$$

Up to the 6.6% discrepancy between $\sin^3(\pi/14)$ and $(\pi-3)/12$, the Wolfenstein $O(\lambda_W^3)$ shifted midpoint reproduces Candidate I. This coincidence ($E_R^{(I)} \approx E_R^{(\text{mid})} + \text{MU} \lambda_W^3$) is a further numerical observation warranting investigation but not constituting a derivation.

6 Summary

6.1 What was proved

1. The midpoint identity. The arithmetic mean of the two Phase 5b candidates from P62 is $E_R^{(\text{mid})} = E_{P1-12} + \text{MU}(\pi+3)/12 = 56.502$, coinciding with the observed E_R to three decimal places. This is a proved numerical identity (Proposition 2.1).

2. The G_2/E_6 bookkeeping. The correction $(\pi+3)/12$ equals $(\pi + N_{\text{gen}})/h^\vee(E_6)$ with $N_{\text{gen}} = 3$. The factor $(\pi+3)$ is the sum of the electron spectral energy $E_e = \pi$ and the number of generations. This is a proved algebraic identity (Proposition 2.3) but not a structural derivation.

3. Wolfenstein corrections do not resolve the degeneracy. The $O(\theta_C^2)$ atmospheric correction is $3.5\times$ the required $\delta = 0.009359$ (Theorem 3.1). No Wolfenstein correction at order $k \leq 3$ matches δ exactly (Corollary 3.7). The Phase 5b degeneracy is not broken by the PMNS sector at these orders.

4. Cubic proximity. The $O(\lambda_W^3)$ term $\text{MU} \sin^3 \theta_C \approx 0.00875$ and $O(\theta_C^3)$ term $\text{MU} \theta_C^3 \approx 0.00897$ bracket the half-splitting $\delta = 0.00936$ at the 4–7% level (Proposition 3.3). This is numerically suggestive of a connection at cubic order but not structurally derivable.

5. The neutrino mass. Assuming the midpoint formula is exact, $m_{\nu_3} \approx 48.7$ meV (Proposition 4.1). This is not an independent prediction because the midpoint formula was derived from the observed Δm_{31}^2 .

6.2 Phase 5b status

Path to closure	What it requires	Status
Candidate I confirmed	Δm_{31}^2 at $\lesssim 0.5\%$ giving $E_R > 56.502$	Pending JUNO/Hyper-K
Candidate II confirmed	Δm_{31}^2 at $\lesssim 0.5\%$ giving $E_R < 56.502$	Pending JUNO/Hyper-K
Midpoint formula proved	Structural derivation of the averaging principle from $J_3(\mathbb{O})$ geometry	Open
Wolfenstein $O(\lambda_W^3)$ resolves it	Exact identity $12 \sin^3(\pi/14) = \pi - 3$	Fails (6.6% off)

Phase 5b is *not yet closed*. The midpoint formula $E_R = E_{\text{Pl}} - 12 + \text{MU}(\pi + 3)/12$ is a compelling conjecture (Conjecture 5.1) supported by a numerical coincidence and a natural bookkeeping identity in G_2/E_6 data. The structural derivation of the averaging principle, and the precision Δm_{31}^2 measurement, remain the two gates to closure.

References

- [1] L. F. Vlegels, *Addendum 53: Type I Seesaw in $J_3(\mathbb{O})$ Language, Cabibbo-Step Dirac Masses, and the Neutrino Mass-Squared Ratio*, TOE Corpus Addendum, May 2026.
- [2] L. F. Vlegels, *Addendum 57: Yukawa Hierarchy from $G_2/\text{SU}(3)$ Orbit Angles*, TOE Corpus Addendum, May 2026.
- [3] L. F. Vlegels, *Addendum 61: The Planck–Seesaw Spectral Gap and the E_6 Dual Coxeter Number*, TOE Corpus Addendum, May 2026.
- [4] L. F. Vlegels, *Addendum 62: G_2/E_6 Threshold Correction to the Planck–Seesaw Gap: Testing $\varepsilon = \text{MU} \cdot \pi/h(G_2)$ and $\varepsilon = \text{MU}/2$ Against the Δm_{31}^2 Observational Window*, TOE Corpus Addendum, May 2026.
- [5] L. F. Vlegels, *Addendum 65: PMNS–Yukawa Compatibility: Quark–Lepton Complementarity and θ_{23} Maximality from G_2 Orbit Geometry*, TOE Corpus Addendum, May 2026.
- [6] L. F. Vlegels, *Addendum 66: Reactor Angle and CP Phase from the G_2 Dihedral Modulus*, TOE Corpus Addendum, May 2026.
- [7] L. F. Vlegels, *Addendum 68: Wolfenstein Expansion of the PMNS Angles to $O(\lambda_W^3)$ from G_2 -Orbit Subleading Corrections*, TOE Corpus Addendum, May 2026.
- [8] Particle Data Group, *Review of Particle Physics*, Chinese Physics C **38** (2024).

- [9] NuFIT 5.3 Collaboration, <http://www.nu-fit.org> (2024); I. Esteban et al., JHEP **2020** (2020) 178.
- [10] JUNO Collaboration, *JUNO Physics and Detector*, Prog. Part. Nucl. Phys. **123** (2022) 103927.
- [11] Hyper-Kamiokande Collaboration, *Hyper-Kamiokande Design Report*, arXiv:1805.04163 (2018).
- [12] L. Wolfenstein, *Parametrization of the Kobayashi-Maskawa Matrix*, Phys. Rev. Lett. **51** (1983) 1945.
- [13] V. G. Kac, *Infinite Dimensional Lie Algebras*, 3rd ed., Cambridge University Press (1990).
- [14] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press (1990).