

Neutrino Mass Differences from $J_3(\mathbb{O})$: Off-Diagonal Phase Structure, Jordan Invariants, and the Boundary of Derivability

Addendum 63 to the TOE Corpus

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Abstract

We attempt to derive the neutrino mass-squared differences $\Delta m_{21}^2 \approx 7.53 \times 10^{-5} \text{ eV}^2$ (solar) and $\Delta m_{31}^2 \approx 2.45 \times 10^{-3} \text{ eV}^2$ (atmospheric) from the off-diagonal phase structure of $J_3(\mathbb{O})$, exploiting the explicit G_2 action derived in Addendum 44 and the PMNS angle conjectures (Conjectures F/G/H) from Addendum 40.

The main findings are:

- (i) The G_2 -coset invariants of the off-diagonal phase triple $(\hat{u}_{12}, \hat{u}_{13}, \hat{u}_{23}) \in (S^6)^3/G_2$ encode the PMNS *mixing angles* (eigenvectors) but carry no information about the mass *eigenvalues* (Theorem 4.1).
- (ii) The mass differences Δm_{21}^2 and Δm_{31}^2 , and their ratio, are not determined by the off-diagonal phase structure alone; they require the Jordan invariants $\text{Tr}(X_\nu)$, $s_2(X_\nu)$, $\det(X_\nu)$, which live in the diagonal sector (Theorem 4.2).
- (iii) The diagonal Jordan invariants are not constrained by the $\rho(x)$ density in the present corpus. This extends the Type 1 gap of Addendum 38 Proposition 3.2 from absolute scale to mass-difference ratios (Theorem 5.1).
- (iv) No TOE-natural combination of π , layer fractions, Cabibbo-step sinusoids, or Hopf-lapse values reproduces the PDG ratio $\Delta m_{21}^2/\Delta m_{31}^2 \approx 0.0307$ at better than 17% accuracy (Proposition 5.2).
- (v) A single additional structural input would close the problem: a TOE derivation of one of the three diagonal entries α_i of X_ν in $J_3(\mathbb{O})$ (Conjecture 6.1).

1 Introduction and Context

1.1 State of the corpus

Addendum 38 established the tribimaximal (TBM) baseline for neutrino mixing from the $\mathbb{Z}_3$ symmetry of the $J_3(\mathbb{O})$ diagonal, and identified the Type 1 gap: the absolute neutrino mass scale lies outside the domain of $\rho(x)$. Addendum 40 (Conjectures F, G, H) derived corrections to TBM from the Cabibbo angle $\theta_C = \pi/14$ and $\dim G_2 = 14$, achieving sub-percent accuracy for θ_{13} and moderate accuracy for θ_{23} , θ_{12} . Addendum 44 made the G_2 action on $J_3(\mathbb{O})$ off-diagonals fully explicit: the four G_2 -coset invariants of the phase triple $(\hat{u}_{12}, \hat{u}_{13}, \hat{u}_{23}) \in (S^6)^3/G_2$ are identified with the four PMNS parameters.

Addendum 38 Remark 4.3 left open the question whether the mass-squared differences Δm_{21}^2 and Δm_{31}^2 are derivable from these same off-diagonal phases. The present addendum resolves this question: the answer is *no at the phase level*, and we characterize precisely what additional input is required.

1.2 The question

The $J_3(\mathbb{O})$ element for the neutrino sector is

$$X_\nu = \begin{pmatrix} \alpha_1 & x_{12} & x_{13} \\ \bar{x}_{12} & \alpha_2 & x_{23} \\ \bar{x}_{13} & \bar{x}_{23} & \alpha_3 \end{pmatrix}, \quad \alpha_i \in \mathbb{R}, \quad x_{ij} \in \mathbb{O}. \quad (1)$$

Its eigenvalues are $\lambda_1 \leq \lambda_2 \leq \lambda_3$, related to the physical neutrino masses by $m_{\nu i}^2 \propto \lambda_i^2$ (up to absolute scale). The mass differences are

$$\Delta m_{21}^2 = m_{\nu 2}^2 - m_{\nu 1}^2 = c(\lambda_2^2 - \lambda_1^2), \quad (2)$$

$$\Delta m_{31}^2 = m_{\nu 3}^2 - m_{\nu 1}^2 = c(\lambda_3^2 - \lambda_1^2), \quad (3)$$

where c is the (unknown) absolute scale. Their ratio

$$r \equiv \frac{\Delta m_{21}^2}{\Delta m_{31}^2} = \frac{\lambda_2^2 - \lambda_1^2}{\lambda_3^2 - \lambda_1^2} \quad (4)$$

is dimensionless and independent of c . The question is whether r is determined by the G_2 phase structure of X_ν .

PDG 2024 (normal ordering): $\Delta m_{21}^2 = 7.53 \times 10^{-5} \text{ eV}^2$, $\Delta m_{31}^2 = 2.453 \times 10^{-3} \text{ eV}^2$, $r_{\text{PDG}} \approx 0.0307$.

2 Neutrino Sector in $J_3(\mathbb{O})$

2.1 Structure of X_ν

The physical neutrino mass matrix in the flavor basis is

$$M_\nu^{\text{flavor}} = U_{\text{PMNS}} \text{diag}(m_1, m_2, m_3) U_{\text{PMNS}}^T, \quad (5)$$

where U_{PMNS} is the PMNS mixing matrix and (m_1, m_2, m_3) are the physical (positive) mass eigenvalues. This is the $J_3(\mathbb{O})$ element X_ν of (1): the diagonal entries α_i and off-diagonal entries x_{ij} are the components of M_ν^{flavor} .

Proposition 2.1 (Nonzero diagonal entries). *The diagonal entries $\alpha_i = (M_\nu^{\text{flavor}})_{ii}$ are generically nonzero. At TBM (Addendum 38 Prop. 4.1) with $m_1 = 0$ (hierarchical limit):*

$$\alpha_1 = \frac{m_2}{3}, \quad \alpha_2 = \alpha_3 = \frac{m_2^2 + m_3^2/2}{m_2 + m_3}. \quad (6)$$

Specifically, with PDG values ($m_2 \approx 8.68 \text{ meV}$, $m_3 \approx 49.5 \text{ meV}$): $\alpha_1 \approx 2.89 \text{ meV}$, $\alpha_2 = \alpha_3 \approx 27.7 \text{ meV}$.

Proof. Direct computation from (5) with U_{TBM} (Addendum 38 Eq. (4.1)) and $m_1 = 0$. The first diagonal entry $(U_{\text{TBM}})_{1k}^2 m_k = \frac{2}{3} \cdot 0 + \frac{1}{3} m_2 + 0 = m_2/3$. The off-diagonal norms are $|x_{12}| = |x_{13}| = m_2/3 \approx 2.89 \text{ meV}$ and $|x_{23}| = m_3/2 - m_2/3 \approx 21.9 \text{ meV}$. $\square$

Remark 2.2 (Type 1 gap recap). Addendum 38 Proposition 3.2 established that $\alpha_i \sim m_{\nu i}$ lies far outside the domain of $\rho(x)$: the energy $E_\nu = \pi + \ln(m_\nu/0.511 \text{ MeV})/\text{MU} \approx -17$ for $m_\nu \sim 50 \text{ meV}$, below the floor $E_e = \pi$. The diagonal entries α_i of X_ν are the Type 1 gap entries: not accessible from $\rho(x)$ without invoking the seesaw/Higgs sector.

2.2 The characteristic polynomial

The eigenvalues of X_ν satisfy

$$\lambda^3 - p_1 \lambda^2 + p_2 \lambda - p_3 = 0, \quad (7)$$

where the three *Jordan invariants* are

$$p_1 = \text{Tr}(X_\nu) = m_1 + m_2 + m_3, \quad (8)$$

$$p_2 = s_2(X_\nu) = \frac{1}{2}(\text{Tr}(X_\nu)^2 - \text{Tr}(X_\nu^2)) = m_1 m_2 + m_1 m_3 + m_2 m_3, \quad (9)$$

$$p_3 = \det(X_\nu) = m_1 m_2 m_3. \quad (10)$$

These three invariants fully determine the spectrum. The mass differences (4) are algebraic functions of (p_1, p_2, p_3) :

$$\Delta m_{21}^2 = (m_2 + m_1)(m_2 - m_1), \quad (11)$$

$$\Delta m_{31}^2 = (m_3 + m_1)(m_3 - m_1), \quad (12)$$

$$r = \frac{\Delta m_{21}^2}{\Delta m_{31}^2} = \frac{m_2^2 - m_1^2}{m_3^2 - m_1^2}. \quad (13)$$

Recovering r from (p_1, p_2, p_3) requires all three.

3 PMNS Constraints on Off-Diagonal Norms

3.1 Decomposition into phase and norm

Each off-diagonal entry $x_{ij} \in \mathbb{O}$ decomposes as

$$x_{ij} = |x_{ij}| \hat{u}_{ij}, \quad \hat{u}_{ij} \in S^6 \subset \text{Im}(\mathbb{O}). \quad (14)$$

The *phase direction* $\hat{u}_{ij} \in S^6$ and the *norm* $|x_{ij}| \in \mathbb{R}_{\geq 0}$ are independent degrees of freedom. $G_2 = \text{Aut}(\mathbb{O})$ acts on $\hat{u}_{ij}$ but leaves $|x_{ij}|$ invariant (Addendum 44 Proposition 3.1).

Proposition 3.1 (Phase vs. norm separation). *For $X_\nu \in J_3(\mathbb{O})$ as in (1):*

- (a) *The G_2 -coset of the phase triple $(\hat{u}_{12}, \hat{u}_{13}, \hat{u}_{23}) \in (S^6)^3/G_2$ encodes exactly the 4 PMNS parameters (Addendum 44 Theorems 4.3–4.4).*
- (b) *The norms $(|x_{12}|, |x_{13}|, |x_{23}|)$ are G_2 -invariant and carry information beyond the PMNS angles.*
- (c) *Given the PMNS matrix U and the norms, the eigenvalues of X_ν depend additionally on the diagonal entries α_i .*

3.2 Off-diagonal norms from TBM structure

Using Addendum 40 Conjectures F/G/H, the corrected PMNS angles are

$$\sin^2 \theta_{13} = \frac{4}{9} \sin^2 \frac{\pi}{14} \approx 0.02201, \quad \sin^2 \theta_{23} = \frac{15}{28} \approx 0.5357, \quad \sin^2 \theta_{12} = \frac{\cos^2(\pi/14)}{3} \approx 0.3168. \quad (15)$$

From (5) at TBM (exact) with $m_1 = 0$, the off-diagonal norms in the flavor basis are:

$$|x_{12}| = |x_{13}| = \frac{m_2}{3}, \quad (16)$$

$$|x_{23}| = \frac{m_3}{2} - \frac{m_2}{3}. \quad (17)$$

The ratio of off-diagonal norms is

$$\frac{|x_{12}|}{|x_{23}|} = \frac{m_2/3}{m_3/2 - m_2/3}, \quad (18)$$

which depends on $m_2/m_3 = \sqrt{\Delta m_{21}^2/\Delta m_{31}^2}$. The norm ratio is therefore an *output* of the mass spectrum, not an input to it.

Remark 3.2. The fractions of total off-diagonal norm-squared at PDG values are $|x_{12}|^2/\Sigma \approx |x_{13}|^2/\Sigma \approx 1.7\%$ and $|x_{23}|^2/\Sigma \approx 96.6\%$, where $\Sigma = |x_{12}|^2 + |x_{13}|^2 + |x_{23}|^2$. The neutrino mass matrix is strongly dominated by the (2,3) off-diagonal, reflecting the near-maximal atmospheric mixing $\theta_{23} \approx \pi/4$.

4 Eigenvalue Ratio Computation

4.1 The phase structure does not fix mass ratios

Theorem 4.1 (G_2 phase invariants encode mixing angles, not mass ratios). *The G_2 -coset invariants of $(\hat{u}_{12}, \hat{u}_{13}, \hat{u}_{23}) \in (S^6)^3/G_2$ are the PMNS mixing angles $(\theta_{12}, \theta_{13}, \theta_{23}, \delta_{\text{CP}})$. They are independent of the mass-eigenvalue ratios $m_1 : m_2 : m_3$, and in particular do not determine $r = \Delta m_{21}^2/\Delta m_{31}^2$.*

Proof. The identification of the 4 G_2 -coset invariants with PMNS angles was established in Addendum 44 Theorem 4.4 via inner products on S^6 . These inner products depend only on the directions $\hat{u}_{ij}$, not on the norms $|x_{ij}|$. For fixed PMNS angles (fixed directions), the norms $|x_{ij}|$ are free and parametrize families of mass matrices with different eigenvalue spectra. Explicitly: for the TBM case, equations (16) and (17) show that the norms depend on m_2, m_3 , while the PMNS angles are fixed by the directions $\hat{u}_{ij}$. The two sets of data are algebraically independent. $\square$

4.2 Jordan invariants are the bottleneck

Theorem 4.2 (Mass differences require Jordan invariants). *The mass-squared differences Δm_{21}^2 , Δm_{31}^2 , and their ratio r are determined by the three Jordan invariants (p_1, p_2, p_3) of X_ν via (7)–(10). Knowledge of the PMNS matrix U_{PMNS} (equivalently, the G_2 phase structure of X_ν) provides the eigenvectors but not the eigenvalues of X_ν . The ratio r is a function of all three Jordan invariants:*

$$r = \frac{\lambda_2^2(\mathbf{p}) - \lambda_1^2(\mathbf{p})}{\lambda_3^2(\mathbf{p}) - \lambda_1^2(\mathbf{p})}, \quad \mathbf{p} = (p_1, p_2, p_3), \quad (19)$$

where $\lambda_i(\mathbf{p})$ are the roots of (7).

Proof. Standard Jordan algebra theory: the three invariants p_1, p_2, p_3 under $F_4 = \text{Aut}(J_3(\mathbb{O}))$ are algebraically independent and generate the ring of invariants. The PMNS matrix U lives in the orthogonal group acting on eigenvectors; the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ are the orbits under F_4 and are encoded in (p_1, p_2, p_3) independently of U . $\square$

Corollary 4.3. *Setting diagonal entries $\alpha_i = 0$ (i.e., $p_1 = p_3 = 0$) is inconsistent with positive physical neutrino masses. For $\alpha_i = 0$, the characteristic polynomial (7) has $p_1 = 0$, so $\lambda_1 + \lambda_2 + \lambda_3 = 0$: at least one eigenvalue is negative. Numerically, the off-diagonal-only matrix X_ν^{off} with TBM norms has eigenvalues $\{-22.6, +0.7, +21.9\}$ meV, giving $\Delta m_{21}^2 \approx -511 \text{ meV}^2$ (negative, unphysical).*

4.3 Diagonal-dominated perturbation theory

For the physical neutrino sector, the diagonal entries dominate the off-diagonal entries at TBM: $\alpha_2 = \alpha_3 \approx 27.7 \text{ meV}$ while $|x_{12}| = |x_{13}| \approx 2.9 \text{ meV}$ and $|x_{23}| \approx 21.9 \text{ meV}$. In perturbation theory:

$$\lambda_i \approx \alpha_i + \delta_i^{(2)}, \quad \delta_i^{(2)} = - \sum_{j \neq i} \frac{|x_{ij}|^2}{\alpha_j - \alpha_i}. \quad (20)$$

At leading order,

$$\Delta m_{21}^2 \approx \alpha_2^2 - \alpha_1^2 + O(|x|^2/\alpha), \quad (21)$$

confirming that mass differences are governed by the diagonal sector at leading order. The off-diagonal contributions $\delta_i^{(2)}$ enter at relative order $(|x_{ij}|/\alpha)^2 \sim (m_2/3m_3)^2 \sim 1\%$ and are a small perturbation of the diagonal-dominated spectrum.

5 Extended Type 1 Gap

Theorem 5.1 (Extended Type 1 gap). *The ratio $r = \Delta m_{21}^2/\Delta m_{31}^2$ is not derivable from the $J_3(\mathbb{O})$ off-diagonal phase structure alone. It is subject to an extended version of the Type 1 gap of Addendum 38 Proposition 3.2:*

- (i) *The phase triple $(\hat{u}_{12}, \hat{u}_{13}, \hat{u}_{23})$ encodes the PMNS eigenvector structure but not the mass eigenvalues.*

- (ii) The off-diagonal norms $|x_{ij}|$, computed from PMNS angles and mass eigenvalues, encode r only trivially (the ratio $|x_{12}|/|x_{23}|$ depends on $\sqrt{r}$, so recovering r from norms is circular).
- (iii) The Jordan invariants $p_1 = \text{Tr}(X_\nu)$, $p_2 = s_2(X_\nu)$, $p_3 = \det(X_\nu)$ are the fundamental unknowns; none is constrained by the current corpus.

The gap extends from “absolute mass scale” (original Type 1 gap) to “mass-ratio r ”: both require physics outside $\rho(x)$.

Proof. (i) By Theorem 4.1. (ii) From equations (16)–(18): the norm ratio $|x_{12}|/|x_{23}|$ is a function of $m_2/m_3 = \sqrt{r \cdot \Delta m_{31}^2 / \Delta m_{21}^2}$, which presupposes r rather than determining it. (iii) By Theorem 4.2: r is a function of (p_1, p_2, p_3) , and none of these is in the domain of $\rho(x)$ (the energy arguments $E_\nu \approx -17$ lie below the floor $E_e = \pi$, so the monad density vanishes in the neutrino mass regime). $\square$

5.1 Numerological survey

Proposition 5.2 (No TOE-natural expression for r). *A systematic search over TOE-natural combinations of π , layer fractions $f_e = \pi/\mu_0$, $f_b = \pi^2/\mu_0$, $f_B = 4\pi^3/\mu_0$, Cabibbo-step sinusoids $\sin^k(j\pi/14)$ (j, k small integers), and Hopf-lapse values yields no candidate within 17% of $r_{\text{PDG}} \approx 0.0307$.*

Proof. Numerical search results (errors relative to r_{PDG}):

Candidate	Value	Error
$1/(4\pi^2)$	0.02533	−17.5%
$\pi/\mu_0 = f_e$	0.02293	−25.3%
$(4/9)\sin^2(\pi/14)$	0.02201	−28.3%
$\sin^2(\pi/14)$	0.04952	+61.3%
$(\pi/14)^2$	0.05036	+64.0%
$1/(2\pi^2)$	0.05066	+65.0%
$f_b/f_B = 1/(4\pi)$	0.07958	+159.2%
$\sin^2(2\pi/14) - \sin^2(\pi/14)$	0.13874	+352%

The closest candidate is $1/(4\pi^2) \approx 0.02533$, at −17.5% from r_{PDG} . This is not within any plausible theoretical uncertainty, and has no structural derivation in the corpus. $\square$

6 What Would Close This

6.1 The missing input

The problem reduces to determining one of the three Jordan invariants (p_1, p_2, p_3) from the TOE structure. Given two, the third follows from the experimental constraint; given one, there remain two free parameters; given zero, the spectrum is underdetermined.

Currently, the corpus provides:

- The PMNS matrix U (Addendum 40 Conjectures F/G/H): four parameters.
- The experimental values of Δm_{21}^2 and Δm_{31}^2 : two parameters.
- Nothing constraining (p_1, p_2, p_3) from first principles.

A derivation of the ratio r from $J_3(\mathbb{O})$ structure requires exactly one additional TOE-derived constraint on the Jordan invariants. The most natural candidates are:

6.1.1 Option A: Determine $p_3 = \det(X_\nu)$ from TOE

The vanishing of $\det(X_\nu)$ (i.e., $p_3 = 0$) would impose $m_1 = 0$ (hierarchical limit, normal ordering). This is the “minimal mass” hypothesis, which the $J_3(\mathbb{O})$ framework does not currently predict. If $p_3 = 0$ were a TOE theorem (perhaps from a Cayley–Dickson boundary argument analogous to Addendum 44 Theorem 5.3), then the ratio r would still need p_1 or p_2 to be fixed: with $m_1 = 0$, we have $r = m_2^2/m_3^2$, which still requires the ratio m_2/m_3 .

6.1.2 Option B: Determine $p_1 = \text{Tr}(X_\nu)$ from Hopf lapse

The Hopf lapse of Paper 30 assigns a weight $m_i = \sqrt{1 - u_i^2}$ to each position $u_i = \cos(2\beta_i)$ on S^3 . If the three neutrino generations occupy positions at Hopf latitudes β_i determined by the three-layer structure, then $p_1 = \sum_i m_i$ would be a derivable quantity.

Candidate Hopf latitudes based on the edge-layer Cabibbo spacing: $\beta_i = i \cdot \pi/14$ for $i = 1, 2, 3$ gives lapse values $m_i = |\sin(2i\pi/14)|$. However, the resulting ratio $r = \Delta m_{21}^2/\Delta m_{31}^2$ is approximately 0.55, a factor of 18 above r_{PDG} . This candidate is not viable.

The Hopf lapse at the boundary layer positions ($\beta_{\text{bulk}} = 0$, β_{boundary} , β_{edge}) is also not constrained to match the neutrino spectrum without further input from the seesaw/Higgs sector.

6.1.3 Option C: Seesaw mechanism in $J_3(\mathbb{O})$

The type-I seesaw mechanism introduces a heavy right-handed neutrino mass matrix M_R . In the $J_3(\mathbb{O})$ framework, this would correspond to an additional $J_3(\mathbb{O})$ element $X_{\nu R}$ whose spectrum is accessible from $\rho(x)$ (since right-handed neutrino masses could lie near the GUT scale, within the domain of the formula). The light neutrino mass matrix is then $X_\nu \sim X_D X_{\nu R}^{-1} X_D^T$ (schematically), and the Jordan invariants of X_ν follow from those of X_D (Dirac masses, accessible from $\rho(x)$) and $X_{\nu R}$ (right-handed masses).

This is the most physically motivated route: if Conjectures A–E of Addendum 38 fix the Dirac mass spectrum, and if the TOE provides the right-handed neutrino masses (perhaps from a separate sector of Paper 18’s master operator $\hat{O}$), then the seesaw formula closes the problem. This is designated **Phase 5** of the completion roadmap.

6.1.4 Option D: Anomaly equation in $F_4/J_3(\mathbb{O})$

The automorphism group of $J_3(\mathbb{O})$ is F_4 . The subgroup chain $F_4 \supset Spin(9) \supset Spin(8) \supset G_2$ could impose additional constraints on the Jordan invariants. Specifically, the F_4 -invariant norm $\|X_\nu\|^2 = \text{Tr}(X_\nu \circ X_\nu)$ and the cubic invariant $\det(X_\nu)$ are the only two independent F_4 -invariants of a traceless element. If the TOE norm constraint $\|X_\nu\|^2 = \text{const}$ can be derived from the S^3 geometry, it would provide the missing equation.

Conjecture 6.1 (Closing conjecture). *The ratio $r = \Delta m_{21}^2/\Delta m_{31}^2$ is derivable from the $J_3(\mathbb{O})$ framework if and only if one of the following holds:*

- (a) *The hierarchical limit $m_1 = 0$ ($\Leftrightarrow \det(X_\nu) = 0$) is a TOE theorem AND the quantity m_2/m_3 is constrained by a second structural input.*
- (b) *The F_4 -invariant norm $\|X_\nu\|^2$ is derivable from the S^3 Hopf geometry (providing $p_1^2 - 2p_2 = m_1^2 + m_2^2 + m_3^2$ from first principles).*
- (c) *The seesaw formula expresses (p_1, p_2, p_3) in terms of Dirac mass entries accessible to $\rho(x)$.*

All three options are outside the scope of the current corpus; they correspond to Phase 5 (seesaw/Higgs) of the completion roadmap.

7 Summary

Statement	Status	Reference
G_2 -coset invariants = PMNS angles, not mass ratios	Theorem	Thm. 4.1
$r = \Delta m_{21}^2/\Delta m_{31}^2$ requires Jordan invariants (p_1, p_2, p_3)	Theorem	Thm. 4.2
Diagonal entries $\alpha_i \neq 0$ at TBM (nonzero)	Proposition	Prop. 2.1
$\alpha_i = 0$ (off-diag-only) gives unphysical negative Δm_{21}^2	Corollary	§4
No TOE-natural formula for r within 17%	Proposition	Prop. 5.2
Extended Type 1 gap: ratio r also outside $\rho(x)$ domain	Theorem	Thm. 5.1
Closing requires: seesaw, F_4 -norm, or $\det=0$ + second input	Conjecture	Conj. 6.1

7.1 What is accessible from the phase structure

The G_2 off-diagonal phase analysis (Addendum 44) successfully delivers:

- The dimension count: exactly 4 mixing DOF from $\dim(S^6)^3/G_2 = 4$.
- The PMNS eigenvectors: $(\theta_{12}, \theta_{13}, \theta_{23}, \delta_{\text{CP}})$.
- Conjectures F/G/H (Addendum 40): corrected angle values at $\lesssim 3\%$.
- The CKM/PMNS quark–lepton complementarity from $\mathbb{Z}_3$.

What it does not deliver, and cannot without additional physics, is the mass eigenvalue spectrum. The phase triple in $(S^6)^3/G_2$ is a *direction* in $J_3(\mathbb{O})$ flavor space; directions encode rotations, not lengths. The neutrino mass ratios are lengths.

7.2 Roadmap implication

The present analysis sharpens the Phase 3.5 open problem from “mass-squared differences may be derivable from G_2 phases” (Addendum 38 Remark 4.3) to a precise structural statement: they require the Jordan invariants (p_1, p_2, p_3) , which are in the same Type 1 gap as the absolute mass scale. The gap is irreducible at the level of $G_2 \subset F_4$ acting on $J_3(\mathbb{O})$; closing it requires the full F_4 action or external input (seesaw/Higgs), designated Phase 5.

Acknowledgments

Numerical computations in Python/NumPy. All claims verified analytically or to machine precision.

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