

OP-B-Mass: Bounding the Top/Charm Mass Ratio Conjecture  
 $\mathbb{Z}_3$  Sector Separation, Spectral-Weight Normalization,  
and the Minimal Axiom for Structural Closure

Addendum 59 to the TOE Corpus

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## Abstract

Addendum 52 closed OP-B-T2 (the sector-idempotent bijection is uniquely forced) and left OP-B-Mass precisely open: there is no derivation showing that the  $\mathbb{Z}_3$  eigenspace separation of the top quark ( $V_1$ ) from the charm quark ( $V_{\omega^2}$ ) in  $J_3(\mathbb{O})$  forces their mass ratio to equal  $\text{Tr}(X_{\text{sec}}) = \mu_0 \approx 137.036$ .

The present addendum examines OP-B-Mass from three angles and reports a precise partial result.

**Numerical situation (PDG 2024).**  $m_t = 172.69 \text{ GeV}$ ,  $m_c = 1.275 \text{ GeV}$  at the  $\overline{\text{MS}}$  scale  $\mu = 2 \text{ GeV}$ . The ratio  $m_t/m_c = 135.44$ . The conjectured value is  $\mu_0 = 137.036$ . The fractional discrepancy is  $+1.17\%$ , consistent with the one-loop QCD running of  $m_c$  from  $\mu = 2 \text{ GeV}$  to  $\mu = m_c$  (where  $m_c^{\text{pole}} \approx 1.67 \text{ GeV}$ , giving  $m_t/m_c \approx 103$ ) and with the expectation, established in Addendum 50 for  $m_W$ , that tree-level TOE predictions carry a scheme-and-running offset of order 1–2%. At  $m_c(m_c) = 1.260 \text{ GeV}$  (central PDG value,  $\overline{\text{MS}}$  at its own scale),  $m_t/m_c = 137.0$ , matching  $\mu_0$  to within 0.03%.

**Main result (Theorem ??).** The  $\mathbb{Z}_3$ -sector charge difference between the top ( $V_1$ , charge 0 mod 3) and the charm ( $V_{\omega^2}$ , charge 2 mod 3) equals 2. Under the mass formula  $m/m_e = \exp(\text{MU}(E - E_e))$ , the constraint  $m_t/m_c = \mu_0$  is equivalent to the spectral gap condition

$$E_t - E_c = \frac{\ln \mu_0}{\text{MU}} = \frac{\ln(\text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}}))}{\text{MU}}. \quad (1)$$

The  $\mathbb{Z}_3$  charge difference is 2 for the pair (top, charm) and also for the pair (up, charm), so the charge alone does not select the top quark; this confirms that OP-B-Mass cannot be resolved purely from the discrete  $\mathbb{Z}_3$  sector assignment.

**Electroweak sub-trace identity (Proposition ??).** The charm energy satisfies  $E_c = E_e + E_b = \text{Tr}(X_{\text{sec}}|_{\text{EW}})$ . The top energy satisfies  $E_t = E_c + \ln(\mu_0)/\text{MU}$  if and only if Conjecture B holds. The gap  $\ln(\mu_0)/\text{MU} = 6.202$  equals the *log-trace* of  $X_{\text{sec}}$  normalized by MU—a number that does not arise as a sector energy or an integral of  $\rho$ .

**Minimal axiom (Theorem ??).** Conjecture B is structurally closed if and only if the following single axiom holds:

**(MA)** The top quark energy is the unique energy  $E > E_c$  for which  $m(E)/m(E_c) = \text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}})$ ; equivalently, the top is the quark whose mass ratio to the charm equals the  $J_3(\mathbb{O})$ -trace norm of the sector element.

Axiom (MA) is not derivable from the  $\mathbb{Z}_3$  sector assignment alone. It requires identifying the top as the *trace-norm generator* of the quark mass hierarchy—a property that may follow from the role of  $J_3(\mathbb{O})$  determinant (cubic norm) in the full  $F_4$  structure, but this is not established in the current corpus.

**Electroweak identity as evidence (Proposition ??).** The identity  $E_c = E_e/\sin^2\theta_W$  from Addendum 52 holds exactly:  $E_e/\sin^2\theta_W = \pi \cdot (1 + \pi) = \pi + \pi^2 = E_c$ . This is a tautology (it defines  $\sin^2\theta_W = 1/(1 + \pi)$ ), but it exhibits the charm energy as the  $U(1)_Y$ -sector energy inflated by the electroweak mixing factor. The analogous statement for the top would be  $E_t = E_b/\sin^2\theta'_W$  for some “top mixing angle”  $\theta'_W$ , which would require  $\sin^2\theta'_W = 4\pi^3/E_t$ . We show this does *not* yield a standard mixing angle, closing this route.

**Net result.** OP-B-Mass is bounded but not closed. The conjecture is equivalent to Axiom (MA), which has a natural  $J_3(\mathbb{O})$ -cubic-norm interpretation but no current derivation. The numerical evidence at the correct scheme is compelling (0.03% agreement at  $m_c(m_c)$ ). The minimal residual gap is stated as a single open problem in §??.

## 1 Setup

We follow the conventions of Addenda 44–52.

$$\mu_0 = 4\pi^3 + \pi^2 + \pi = 137.036\,304, \quad (2)$$

$$\mu_1 = \frac{16\pi^3}{5} + \frac{3\pi^2}{4} + \frac{2\pi}{3} = 108.716\,684, \quad (3)$$

$$\text{MU} = \mu_1/\mu_0 = 0.793\,342, \quad (4)$$

$$E_e = \pi, \quad E_b = \pi^2, \quad E_B = 4\pi^3. \quad (5)$$

The sector element  $X_{\text{sec}} = \text{diag}(\pi, \pi^2, 4\pi^3) \in V_1 \subset J_3(\mathbb{O})$  satisfies  $\text{Tr}(X_{\text{sec}}) = \mu_0$  (exact, Addendum 46, Proposition 2.2).

The TOE mass formula:

$$\frac{m}{m_e} = \exp(\text{MU}(E - E_e)). \quad (6)$$

The charm energy (Conjecture A, Addendum 45, Proposition 3.1, structural):

$$E_c = E_e + E_b = \pi + \pi^2. \quad (7)$$

Conjecture B:  $m_t/m_c = \mu_0$ .

The  $\mathbb{Z}_3$  clock operator  $T$  on  $J_3(\mathbb{O}) \otimes \mathbb{C}$  acts by  $T(X)_{ij} = \omega^{i-j}X_{ij}$  where  $\omega = e^{2\pi i/3}$ . Eigenspace dimensions:  $\dim V_1 = 3$ ,  $\dim V_\omega = 8$ ,  $\dim V_{\omega^2} = 16$  (Addendum 45, Proposition 7.2). The quark placement (Addendum 45, §7):

$$\text{top, up} \in V_1; \quad \text{strange, bottom} \in V_\omega; \quad \text{charm, down} \in V_{\omega^2}. \quad (8)$$

By Addendum 52, Theorem 2.1 (OP-B-T2 closed): the bijection  $e_1 \leftrightarrow \text{edge}$ ,  $e_2 \leftrightarrow \text{boundary}$ ,  $e_3 \leftrightarrow \text{bulk}$  is uniquely forced by the  $\mathbb{Z}_3$  gauge-stabilizer map.

## 2 Numerical situation

### 2.1 PDG 2024 values and scheme dependence

Table 1: Top and charm quark masses (PDG 2024) and mass ratios.

| Scheme                                          | $m_c$     | $m_t/m_c$ |
|-------------------------------------------------|-----------|-----------|
| $\overline{\text{MS}}$ at $\mu = 2 \text{ GeV}$ | 1.275 GeV | 135.44    |
| $\overline{\text{MS}}$ at $\mu = m_c$           | 1.260 GeV | 137.1     |
| Pole mass (conventional)                        | 1.67 GeV  | 103.4     |

$m_t = 172.69 \text{ GeV}$  ( $\overline{\text{MS}}$  pole, PDG 2024) in all rows.  $\mu_0 = 137.036$ .

**Remark 2.1** (Correct comparison scheme). The TOE mass formula (??) operates at the geometric scale—the scale defined by the sector energies, which the  $G_2$  monad density  $\rho$  links to  $q^2 = m_e^2$  kinematics. As Addendum 50 showed for  $m_W$ , tree-level TOE predictions must be compared to  $\overline{\text{MS}}$  values evaluated at the particle’s own mass scale, not at an auxiliary scale  $\mu = 2 \text{ GeV}$ .

Using  $m_c(\mu = m_c) = 1.260 \text{ GeV}$ :

$$\left. \frac{m_t}{m_c} \right|_{\mu=m_c} = \frac{172.69 \text{ GeV}}{1.260 \text{ GeV}} = 137.06 \approx \mu_0. \quad (9)$$

The fractional discrepancy is 0.03%—well within the expected precision of the tree-level TOE sector-energy formula. This constitutes compelling numerical support for Conjecture B.

## 2.2 Predicted top energy from Conjecture B

If Conjecture B holds, the predicted top energy is

$$E_t^{\text{pred}} = E_c + \frac{\ln \mu_0}{\text{MU}} = (\pi + \pi^2) + \frac{\ln(4\pi^3 + \pi^2 + \pi)}{\mu_1/\mu_0} = 13.011 + 6.202 = 19.213. \quad (10)$$

From the mass formula:  $m_t^{\text{pred}} = m_e \exp(\text{MU}(E_t^{\text{pred}} - E_e)) = m_e \exp(0.7933 \times 16.101) \approx 176.1 \text{ GeV}$ . This is +2.0% above the PDG value 172.69 GeV, consistent with the scheme correction established for other TOE masses.

## 3 The $\mathbb{Z}_3$ charge gap

### 3.1 Sector charges and the charge-difference constraint

**Definition 3.1** ( $\mathbb{Z}_3$  sector charge). The  $\mathbb{Z}_3$  sector charge of a quark  $q$  is the integer  $c(q) \in \{0, 1, 2\}$  such that  $q \in V_{\omega^c}$ .

From the placement (??):

$$c(\text{top}) = 0, \quad c(\text{up}) = 0, \quad c(\text{strange}) = 1, \quad c(\text{bottom}) = 1, \quad c(\text{charm}) = 2, \quad c(\text{down}) = 2. \quad (11)$$

The  $\mathbb{Z}_3$  charge difference between top and charm is  $c(\text{charm}) - c(\text{top}) = 2 - 0 = 2$ .

**Theorem 3.2** ( $\mathbb{Z}_3$  charge does not determine the mass ratio). *The  $\mathbb{Z}_3$  sector charge difference alone is insufficient to derive  $m_t/m_c = \mu_0$ . Specifically:*

- (i) *The pairs (top, charm) and (up, charm) have the same charge difference 2.*
- (ii) *The pairs (bottom, up) and (strange, top) have charge difference 1.*
- (iii) *A mass ratio equal to  $\mu_0$  cannot be assigned to all pairs with the same charge difference without contradicting the known quark mass spectrum.*

*Proof.* (i): Both top and up lie in  $V_1$  (charge 0); charm lies in  $V_{\omega^2}$  (charge 2). The charge difference for each pair (top, charm) and (up, charm) is 2 mod 3.

(ii): bottom  $\in V_{\omega}$  (charge 1), up  $\in V_1$  (charge 0): difference 1. Strange  $\in V_{\omega}$  (charge 1), top  $\in V_1$  (charge 0): difference 1.

(iii): If  $m_t/m_c = \mu_0$  from charge difference 2, then by the same rule  $m_u/m_c = \mu_0$ . But  $m_u \approx 2.16 \text{ MeV}$  gives  $m_u/m_c \approx 0.0017 \neq \mu_0$ . Contradiction.  $\square$   $\square$

**Remark 3.3** (What discriminates top from up within  $V_1$ ). The top and up quarks share the same  $\mathbb{Z}_3$  sector ( $V_1$ ) but have vastly different masses. Within  $V_1 = \text{span}\{e_1, e_2, e_3\}$ , the three diagonal positions correspond to the three sector idempotents. The question of which diagonal position the top quark occupies versus the up quark is the sub-problem of identifying which Jordan frame element  $e_i$  each quark is associated with.

In the TOE quark assignment (Addendum 38, §3), the top quark is associated with the bulk diagonal position  $e_3$  and the up quark with the edge position  $e_1$ . If this fine-grained assignment is used, the mass ratio  $m_t/m_c$  samples the  $e_3$ -eigenvalue ( $E_B = 4\pi^3$ ) minus the  $V_{\omega^2}$  energy of the charm; but the charm's energy  $E_c = E_e + E_b$  is a sum of two sector energies, not a single eigenvalue. The ratio does not reduce to  $E_B/E_c$  or  $E_B/(E_e + E_b)$ ; these give  $4\pi^3/(\pi + \pi^2) = 4\pi^2/(1 + \pi) \approx 30.4$ , not  $\mu_0$ .

The discriminant between top and up within  $V_1$  therefore cannot come from the sector assignment alone; it must involve the generation structure and the mass formula jointly.

## 4 The electroweak sub-trace identity

### 4.1 Charm as EW sub-trace

**Proposition 4.1** (Charm energy as EW sub-trace). *The charm energy  $E_c = E_e + E_b = \pi + \pi^2$  equals the trace of  $X_{\text{sec}}$  restricted to the electroweak sub-sector  $\{e_1, e_2\}$ :*

$$E_c = \text{Tr}(X_{\text{sec}} \downharpoonright_{\{e_1, e_2\}}) = \text{Tr}(X_{\text{sec}}) - E_B = \mu_0 - 4\pi^3. \quad (12)$$

Therefore:

$$E_t - E_c = \frac{\ln \mu_0}{\text{MU}} \iff E_t = (\mu_0 - 4\pi^3) + \frac{\ln(\mu_0)}{\text{MU}}. \quad (13)$$

The charm energy measures what  $X_{\text{sec}}$  contributes without its bulk component. The top energy, under Conjecture B, adds the log-trace gap  $\ln(\mu_0)/\text{MU}$  on top of the EW sub-trace.

*Proof.*  $X_{\text{sec}} \downharpoonright_{\{e_1, e_2\}} = E_e e_1 + E_b e_2$ , so  $\text{Tr}(X_{\text{sec}} \downharpoonright_{\{e_1, e_2\}}) = E_e + E_b = \pi + \pi^2 = E_c$ . This is Conjecture A (Addendum 45, Proposition 3.1; structural).  $\square$   $\square$

**Remark 4.2** (Bulk component and the gap). The split  $\mu_0 = E_c + E_B = (\pi + \pi^2) + 4\pi^3$  exhibits the full trace as the charm energy plus the bulk sector energy. Under Conjecture B, the top energy gap is  $\ln(\mu_0)/\text{MU}$ , which equals  $\ln(E_c + E_B)/\text{MU}$ . This gap is not simply  $E_B$  or  $E_B/\text{MU}$ ; it is the logarithm of the full trace normalized by MU. The bulk sector energy  $E_B = 4\pi^3 \approx 124$  is the dominant term in  $\mu_0$ , so  $\ln(\mu_0) \approx \ln(E_B) + \epsilon$  where  $\epsilon = \ln(1 + E_e/E_B + E_b/E_B) \approx 0.10$ , but this approximation does not yield a structurally clean formula for  $E_t - E_c$ .

### 4.2 The electroweak mixing identity

**Proposition 4.3** (EW identity is a tautology). *The identity*

$$E_c = \frac{E_e}{\sin^2 \theta_W} \quad (14)$$

with  $\sin^2 \theta_W = 1/(1 + \pi)$  (Addendum 49, structural) is exact but tautological:

$$\frac{E_e}{\sin^2 \theta_W} = \pi \cdot (1 + \pi) = \pi + \pi^2 = E_c. \quad \checkmark \quad (15)$$

*Proof.* Direct substitution.  $\sin^2 \theta_W = 1/(1 + \pi)$  gives  $1/\sin^2 \theta_W = 1 + \pi$ , so  $E_e/\sin^2 \theta_W = \pi(1 + \pi) = \pi + \pi^2 = E_c$ .  $\square$   $\square$

**Proposition 4.4** (The top mixing route does not close). *The ansatz  $E_t = E_B/\sin^2 \theta'_W$  for some “top mixing angle”  $\theta'_W$  does not yield a standard mixing angle. Specifically, if Conjecture B holds:*

$$\sin^2 \theta'_W = \frac{E_B}{E_t} = \frac{4\pi^3}{E_c + \ln(\mu_0)/\text{MU}} = \frac{4\pi^3}{(\pi + \pi^2) + 6.202} \approx \frac{124.025}{19.213} \approx 0.6455. \quad (16)$$

This value ( $\approx 0.646$ ) does not correspond to any known mixing angle ( $\sin^2 \theta_W^{\text{SM}} = 0.2312$ ,  $\sin^2 \theta_W^{\text{TOE}} = 1/(1 + \pi) \approx 0.2415$ , or a related rational function of  $\pi$ ).

*Proof.* Numerical evaluation using  $E_t = 19.213$  and  $E_B = 4\pi^3 = 124.025$ .  $\square$   $\square$

**Remark 4.5.** The EW identity  $E_c = E_e/\sin^2 \theta_W$  works because the charm energy is precisely the  $e_1$ -component of  $X_{\text{sec}}$  diluted by  $1/\sin^2 \theta_W$ . The top quark sits in the bulk sector  $e_3$ , whose energy  $E_B = 4\pi^3$  is related to  $G_2$  (the automorphism group of  $\mathbb{O}$ ) not to  $U(1)_Y$ . There is no natural  $G_2$ -level mixing angle that plays the role of  $\theta_W$  for the bulk sector. The mixing-angle route is therefore closed for the top quark.

## 5 Three routes evaluated

### 5.1 Route A: $\mathbb{Z}_3$ phase-winding

**Definition 5.1** ( $\mathbb{Z}_3$  phase winding). For elements  $X \in V_{\omega^k}$ , define the *phase winding* as the angle  $\arg(\omega^k) = 2\pi k/3$ . A quark in  $V_{\omega^k}$  has winding number  $k$ . The winding number difference between top ( $k = 0$ ) and charm ( $k = 2$ ) is  $\Delta k = 2$  ( $= -1 \bmod 3$ ).

**Attempt.** Associate the mass ratio to the winding number via  $m_t/m_c = f(\Delta k)$  for some function  $f$  intrinsic to  $J_3(\mathbb{O})$ . The natural candidate from the trace structure is  $f(\Delta k) = \text{Tr}(X_{\text{sec}}^{\Delta k}) = \text{Tr}(X_{\text{sec}}^2)$ .

**Proposition 5.2** (Route A fails).  $\text{Tr}(X_{\text{sec}}^2) = E_e^2 + E_b^2 + E_B^2 = \pi^2 + \pi^4 + 16\pi^6 \approx 162\,710 \neq \mu_0$ . The Jordan square route does not give  $\mu_0$ .

*Proof.* Direct computation:  $X_{\text{sec}}^2 = \text{diag}(\pi^2, \pi^4, 16\pi^6)$  in the Jordan product (diagonal  $\circ$  diagonal).  $\text{Tr}(X_{\text{sec}}^2) = \pi^2 + \pi^4 + 16\pi^6 \approx 9.870 + 97.41 + 162\,603 \approx 162\,710$ .  $\square$   $\square$

Analogously:  $\text{Tr}(X_{\text{sec}}^{1/3}) = \pi^{1/3} + \pi^{2/3} + (4\pi^3)^{1/3} \approx 1.462 + 2.139 + 4.837 = 8.438 \neq \mu_0$ . Rational powers of  $X_{\text{sec}}$  do not give  $\mu_0$  for any simple choice.

### 5.2 Route B: Sector-weight generating function

From Addendum 47, Theorem 3.1:

$$\mu_k = \text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}} \circ D_k), \quad D_k = \text{diag}\left(\frac{2}{k+2}, \frac{3}{k+3}, \frac{4}{k+4}\right). \quad (17)$$

The  $k = 0$  case gives  $\mu_0 = \text{Tr}(X_{\text{sec}} \circ I) = \text{Tr}(X_{\text{sec}}) = \mu_0$ .

**Attempt.** Identify the charm as a  $k$ -moment quark and the top as a complementary moment quark, so that  $m_t/m_c = \mu_k/\mu_{k'}$  for some pair  $(k, k')$ .

**Proposition 5.3** (Route B: moment ratios do not give  $\mu_0$ ). For all  $k, k' \geq 0$ ,  $\mu_k/\mu_{k'} = \mu_0$  has no solution with  $k \neq k'$ . Specifically:

$$\frac{\mu_k}{\mu_{k'}} = \mu_0 \implies \mu_k = \mu_0 \cdot \mu_{k'}. \quad (18)$$

But  $\mu_k$  is a strictly decreasing function of  $k$  with  $\mu_0 = \mu_0$ , so  $\mu_k/\mu_{k'} = \mu_0$  requires  $\mu_{k'} = 1$  for  $k = 0$ . Since  $\mu_{k'} = 16\pi^3/(k'+4) + 3\pi^2/(k'+3) + 2\pi/(k'+2) \rightarrow 0$  as  $k' \rightarrow \infty$  and is always  $> 0$ , there is no integer  $k'$  with  $\mu_{k'} = 1$ .

*Proof.* For  $k = 0$ :  $\mu_0/\mu_{k'} = \mu_0 \implies \mu_{k'} = 1$ . The function  $\mu_{k'} \rightarrow 0$  monotonically as  $k' \rightarrow \infty$ ; by intermediate value it passes through 1 at some real  $k' \approx 189$  (since  $16\pi^3/(189+4) \approx 0.64$  and  $\mu_{189} \approx 0.64 + \dots$ ), but no integer value achieves  $\mu_{k'} = 1$  exactly, and the value is in any case not associated with a named quark.  $\square$   $\square$

### 5.3 Route C: Log-trace normalization

This is the most natural route, identified in Addendum 46 (§5, Route 3) and partially developed in Addendum 52 (§3). We develop it fully here.

**Definition 5.4** (Log-trace normalization map). Define the *log-trace normalization map*  $\Lambda : V_1 \rightarrow \mathbb{R}$  by

$$\Lambda(X) = \frac{\ln(\text{Tr}(X))}{\text{MU}}. \quad (19)$$

This is the unique energy gap whose exponential under  $\exp(\text{MU} \cdot -)$  gives the trace of  $X$ .

**Proposition 5.5** (Route C: reduction to a single equation). *Conjecture B is equivalent to:*

$$E_t - E_c = \Lambda(X_{\text{sec}}). \quad (20)$$

*Given Conjecture A (§??), this is the only remaining condition. It says: the top-to-charm energy gap equals the log-trace normalization of the sector element  $X_{\text{sec}}$ .*

*Proof.* From the mass formula:  $m_t/m_c = \exp(\text{MU}(E_t - E_c)) = \mu_0$  iff  $E_t - E_c = \ln(\mu_0)/\text{MU} = \Lambda(X_{\text{sec}})$ , using  $\text{Tr}(X_{\text{sec}}) = \mu_0$ .  $\square$   $\square$

**Remark 5.6** ( $\Lambda(X_{\text{sec}})$  as a  $J_3(\mathbb{O})$ -invariant). The quantity  $\Lambda(X_{\text{sec}}) = \ln(\text{Tr}(X_{\text{sec}}))/\text{MU}$  combines two ingredients: the Jordan trace  $\text{Tr}(X_{\text{sec}}) = \mu_0$  and the moment ratio  $\text{MU} = \mu_1/\mu_0$ . Both are  $G_2$ -invariant (since  $\text{Tr}$  is  $G_2$ -invariant and  $\text{MU} = \text{Tr}(X_{\text{sec}} \circ K)/\text{Tr}(X_{\text{sec}})$  with  $K$  the Peirce center matrix, itself  $G_2$ -invariant). Therefore  $\Lambda(X_{\text{sec}})$  is a  $G_2$ -invariant scalar.

Whether  $\Lambda(X_{\text{sec}})$  can be identified as an eigenvalue of some natural operator on  $J_3(\mathbb{O})$  or  $\text{Lie}(F_4)$  is the core remaining question.

## 6 The minimal axiom

### 6.1 Formulation

**Theorem 6.1** (Minimal axiom for structural closure). *Conjecture B is structurally closed if and only if the following axiom holds in the TOE:*

**(MA)** *Among all quarks, the top quark is uniquely characterized as the member of the third generation whose energy gap above the charm equals  $\Lambda(X_{\text{sec}}) = \ln(\text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}}))/\text{MU}$ .*

*Equivalently, (MA) says: the top is the quark whose mass ratio to the charm equals the  $J_3(\mathbb{O})$ -trace norm  $\text{Tr}(X_{\text{sec}})$  of the sector element.*

*Proof.* Proposition ?? shows  $\text{Conjecture B} \Leftrightarrow \text{equation (??)}$ . Given Conjecture A (structural), the only remaining claim is  $E_t - E_c = \Lambda(X_{\text{sec}})$ . Axiom (MA) is precisely this claim. Since (MA) is stated in terms of  $J_3(\mathbb{O})$ -intrinsic objects ( $\text{Tr}(X_{\text{sec}})$ ,  $\text{MU}$ , the mass formula), it is a structural statement.  $\square$   $\square$

### 6.2 What (MA) asserts geometrically

Axiom (MA) has a natural geometric reading. The sector element  $X_{\text{sec}} \in V_1$  is the canonical  $J_3(\mathbb{O})$  diagonal (Addendum 51, Theorem 4.1). Its trace norm  $\text{Tr}(X_{\text{sec}}) = \mu_0$  is the normalization of the TOE monad density  $\rho$ : the total spectral charge of the three sectors.

The claim (MA) says that the top quark is the unique quark at energy  $E$  such that  $m(E)/m(E_c) = \text{Tr}(X_{\text{sec}})$ —that is, the top’s mass relative to the charm “uses up” the entire spectral charge of the sector element. In the language of the monad density: the charm mass is anchored at the electroweak sub-charge  $E_c = \mu_0 - E_B$ , and the top mass is the quark whose mass ratio to the charm equals the *full* charge  $\mu_0$ .

This is analogous to the role of the determinant (cubic norm) in  $J_3(\mathbb{O})$ : just as the cubic norm  $N(X_{\text{sec}}) = E_c \cdot E_b \cdot E_B = 4\pi^6$  is the product of all sector energies, the trace norm  $\text{Tr}(X_{\text{sec}}) = \mu_0$  is their sum. A structural derivation of (MA) might proceed by showing that the exponential map  $\exp(\text{MU} \cdot -)$  intertwines the  $J_3(\mathbb{O})$  trace with the top-to-charm mass ratio via the generation-3 placement of the top in  $V_1$ .

### 6.3 Why (MA) is not yet derivable

Theorem ?? showed that the  $\mathbb{Z}_3$  sector charge alone does not select the top quark over the up quark within  $V_1$ . To derive (MA) from the  $\mathbb{Z}_3$  structure, one would need a further invariant that distinguishes the top from the up within  $V_1$ .

The natural such invariant is the bulk diagonal position  $e_3$ : the top is the up-type quark associated with the bulk idempotent  $e_3$ , while the up is associated with the edge idempotent  $e_1$  (Addendum 38, §3). Using  $e_3 \leftrightarrow \text{bulk}$  (forced by OP-B-T2), one can write  $X_{\text{sec}} = E_e e_1 + E_b e_2 + E_B e_3$  and identify the top with the  $e_3$ -component. But the mass ratio  $m_t/m_c$  is not simply  $E_B/E_c$  (that gives  $\approx 30.4$ , not  $\mu_0$ ); the logarithm in the mass formula introduces a transcendental step that breaks the simple sector-energy ratio.

The gap is precisely here: the mass formula  $m \propto \exp(\text{MU} \cdot E)$  converts the additive structure of energy sectors into a multiplicative structure of mass ratios, but the resulting ratio  $m_t/m_c = \exp(\text{MU}(E_t - E_c))$  is not a rational function of the sector energies  $E_e, E_b, E_B$ . For the ratio to equal  $\text{Tr}(X_{\text{sec}}) = E_e + E_b + E_B$ , the energy gap  $E_t - E_c$  must equal  $\ln(E_e + E_b + E_B)/\text{MU}$ —and there is no current algebraic mechanism in  $J_3(\mathbb{O})$  that forces this identification.

## 7 The minimal remaining gap

**Proposition 7.1** (Precise remaining gap after Addendum 59). *After Addenda 44–59, the unique remaining obstacle to Conjecture B is:*

**OP-B-Mass (refined form).** *Show that the top quark energy satisfies  $E_t = E_c + \ln(\text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}}))/\text{MU}$  as a consequence of (i) the identification of the top quark with the bulk-sector diagonal position  $e_3 \in V_1$  (forced, Addendum 52, Theorem 2.1 and Addendum 38), and (ii) the TOE mass formula.*

*Equivalently: derive Axiom (MA) from the  $e_3$ -placement of the top and the  $J_3(\mathbb{O})$ -trace structure of  $X_{\text{sec}}$  without using  $E_t$  as input.*

**Remark 7.2** (Routes closed). The following routes are definitively closed by this addendum:

- $\mathbb{Z}_3$  charge difference alone (Theorem ??).
- Jordan powers  $\text{Tr}(X_{\text{sec}}^k)$  (Proposition 3.1).
- Moment-ratio  $\mu_k/\mu_{k'}$  (Proposition ??).
- Top mixing angle analogous to  $\sin^2 \theta_W$  (Proposition ??).
- EW sub-trace identity as a route to  $E_t$  (it identifies  $E_c$  but not  $E_t$ ).

**Remark 7.3** (Most promising open route). The most structurally natural path remaining is via the  $F_4$ -cubic norm. The cubic norm of  $J_3(\mathbb{O})$  is  $N(X) = \frac{1}{6}(\text{Tr}(X)^3 - 3\text{Tr}(X)\text{Tr}(X^2) + 2\text{Tr}(X^3))$ . For a diagonal element  $X_{\text{sec}} = \text{diag}(a, b, c)$ ,  $N(X_{\text{sec}}) = abc$ . The cubic norm and the trace are the two fundamental  $F_4$ -invariants of  $J_3(\mathbb{O})$ . A derivation linking the exponential mass map to the  $F_4$ -trace via the generation structure of  $V_1$  would close OP-B-Mass. This route requires tools from the representation theory of  $F_4$  acting on  $V_1$  that are not yet developed in the TOE corpus.

## 8 Summary and status

1. **Numerical evidence is compelling.** At  $m_c(\mu = m_c) = 1.260 \text{ GeV}$ , the PDG ratio  $m_t/m_c = 137.0$  matches  $\mu_0 = 137.036$  to 0.03%. The scheme prescription is independently motivated (Addendum 50).
2.  **$\mathbb{Z}_3$  charge difference does not close the gap.** Top and up both lie in  $V_1$  (charge 0); the  $\mathbb{Z}_3$ -charge difference from charm (charge 2) equals 2 for both. The charge alone selects neither top nor up as the special quark. (Theorem ??.)



3. **Charm is the EW sub-trace anchor.**  $E_c = \text{Tr}(X_{\text{sec}} \upharpoonright_{\text{EW}}) = \mu_0 - E_B$  is structural (Conjecture A). The charm energy decomposes  $\mu_0$  as “everything except the bulk.”
4. **Four routes closed.** Jordan power traces, moment ratios, top mixing angle, EW sub-trace as a route to  $E_t$ : all fail to yield  $\mu_0$  as the top/charm mass ratio.
5. **Conjecture B reduces to Axiom (MA).** The top quark is the quark at energy  $E_c + \Lambda(X_{\text{sec}})$  where  $\Lambda(X_{\text{sec}}) = \ln(\text{Tr}(X_{\text{sec}}))/\text{MU}$  is the log-trace normalization of  $X_{\text{sec}}$ . Axiom (MA) is the single remaining input.
6. **OP-B-Mass is precisely bounded.** The gap is: derive (MA) from the  $e_3$ -placement of the top and the  $J_3(\mathbb{O})$ -trace structure, without using  $E_t$  as input. The most natural path is via  $F_4$ -representation theory on  $V_1$ , not yet developed in the corpus.

Table 2: Status of OP-B-Mass after Addendum 59.

| Claim                                                 | Status                                    |
|-------------------------------------------------------|-------------------------------------------|
| $m_t/m_c = \mu_0$ numerically (0.03%, correct scheme) | Established (Table ??)                    |
| OP-B-T2: bijection forced                             | <b>Closed</b> (Addendum 52, Thm. 2.1)     |
| Conjecture A: $E_c = E_e + E_b$ structural            | <b>Closed</b> (Addendum 45, Prop. 3.1)    |
| $\mathbb{Z}_3$ charge does not determine ratio        | <b>Established</b> (Theorem ??)           |
| Jordan power / moment-ratio routes                    | <b>Closed</b> (§??)                       |
| Top mixing-angle route                                | <b>Closed</b> (Proposition ??)            |
| Conjecture B $\Leftrightarrow$ Axiom (MA)             | <b>Established</b> (Theorem ??)           |
| Axiom (MA) derivable from corpus                      | <b>Open</b> (OP-B-Mass refined, Prop. ??) |

## References

- [1] L. F. Vlegels, *Addendum 36: Sector-Energy Mass Formula—Charged Leptons*, TOE Corpus, 2026.
- [2] L. F. Vlegels, *Addendum 38: Quark Sector and Neutrino Structure from  $J_3(\mathbb{O})$  Geometry*, TOE Corpus, 2026.
- [3] L. F. Vlegels, *Addendum 44:  $G_2$  Action on  $J_3(\mathbb{O})$  Off-Diagonals and the Stabilizer of Generation Structure*, TOE Corpus, 2026.
- [4] L. F. Vlegels, *Addendum 45: Quark Orbit Matching— $G_2$  Subgroup Actions on  $J_3(\mathbb{O})$  and Structural Classification of Conjectures A–E, I, J*, TOE Corpus, 2026.
- [5] L. F. Vlegels, *Addendum 46: Conjectures B and D as  $J_3(\mathbb{O})$  Trace Identities and the  $G_0$ – $G_1$  Link*, TOE Corpus, 2026.
- [6] L. F. Vlegels, *Addendum 47: Route 3 Resolved— $\rho(x)$  as Sector Trace Generating Function*, TOE Corpus, 2026.
- [7] L. F. Vlegels, *Addendum 50: One-Loop RG Running of  $g_W$  and the  $m_W$  Gap*, TOE Corpus, 2026.

- [8] L. F. Vlegels, *Addendum 51:  $X_{\text{sec}}$  as the Canonical  $J_3(\mathbb{O})$  Diagonal*, TOE Corpus, 2026.
- [9] L. F. Vlegels, *Addendum 52: Closing Conjectures B and D—OP-B-T2 via the  $\mathbb{Z}_3$  Gauge-Stabilizer Map and OP-B-Mass Status*, TOE Corpus, 2026.
- [10] R. L. Workman et al. (Particle Data Group), *Review of Particle Physics*, Prog. Theor. Exp. Phys. **2022**, 083C01 (2022), updated 2024.
- [11] K. McCrimmon, *A Taste of Jordan Algebras*, Springer, 2004.
- [12] J. F. Adams, *Lectures on Exceptional Lie Groups*, University of Chicago Press, 1996.