

Deriving the Seesaw Scale M_R from the E_6 Adjoint **78** Sector: Five Options, One Hit, and the Unit-Yukawa Identification

Addendum 55 to the TOE Corpus

L. F. Vlegels
Independent Researcher
axiomofchoicepuzzle@gmail.com

May 2026

Contents

1	Setup and notation	2
1.1	The target	2
1.2	TOE constants used throughout	2
1.3	The E_6 adjoint decomposition	3
2	The seesaw scale in TOE energy coordinates	3
3	Five options explored	3
3.1	Option A: Adjoint branching ratio times M_{Pl}	3
3.2	Option B: Spectral-moment combinations	4
3.3	Option C: One-loop RG running of α_s to nonperturbative $\text{SU}(3)_R$	4
3.4	Option D: Killing form Casimir ratios on the $(1, 1, 8)$ block	4
3.5	Option E: Type I seesaw identity with unit Yukawa	5
3.5.1	The formula	5
3.5.2	Numerical result	5
3.5.3	Status of this result	5
4	Option comparison and the TOE energy structure	6
5	Structural interpretation of Option E	6
5.1	Why $m_{D,3} = v_{\text{EW}}$ in E_6 trinification	6
5.2	The seesaw scale as an energy reflection	7
5.3	What the E_6 adjoint sector contributes	7
6	Honest assessment and what remains open	7
6.1	The result	7
6.2	What new formalism is needed	7
6.3	Summary table of open problems	8
7	Summary	8

Abstract

Addendum 53 (Phase 5b) left open the derivation of the right-handed Majorana neutrino mass $M_R \approx 1.24 \times 10^{15}$ GeV from the E_6 adjoint **78** sector. The corresponding TOE energy $E_R = \pi + \ln(M_R/m_e)/\text{MU} \approx 56.5$ lies in a gap with no obvious parent constant. We systematically evaluate five derivation strategies — (A) adjoint branching-ratio times M_{Pl} , (B) spectral-moment combinations, (C) one-loop RG running of α_s , (D) Killing-form Casimir ratios from the $(1, 1, 8)$ block, and (E) the Type I seesaw identity with a unit Yukawa. Options A, C and D miss the target by 30–80%. Option B yields two near-misses: $\mu_1/2 \approx 54.36$ (3.8% low) and $\mu_1/2 + \pi \text{MU} \approx 56.85$ (0.6% high), but neither has a clean structural derivation. Option E — the seesaw identity $M_R = v_{\text{EW}}^2/m_\nu$ with $m_D = v_{\text{EW}}$ (unit Yukawa $y_\nu = 1$) and $m_\nu = \sqrt{\Delta m_{31}^2}$ — reproduces $E_R = 56.486$ with an error of -0.03% . This is not a prediction but a consistency check: the E6 trinification structure forces $y_\nu = 1$ at the unification boundary ($y_\nu = y_t$ in the top-colour sector), and the seesaw then gives $M_R = 1.225 \times 10^{15}$ GeV, within 1.3% of the target. The E_6 adjoint does not yield M_R as a standalone TOE formula; the scale is emergent from the interplay of two sector energies — the electroweak VEV ($E_v \approx 19.64$) and the light neutrino mass ($E_\nu \approx -17.23$) — via the seesaw relation $E_R = 2E_v - E_\nu$. Closing the derivation requires either (i) a TOE formula for m_ν independent of Δm_{31}^2 , or (ii) a derivation of the $\text{SU}(3)_R$ -breaking VEV v_R from the $(1, 1, 8)$ Killing form, both of which remain open.

1 Setup and notation

1.1 The target

Addendum 53, Proposition 3.3 established that the seesaw scale required to reproduce the observed $\Delta m_{31}^2 = 2.45 \times 10^{-3} \text{ eV}^2$ with third-generation Dirac mass equal to the electroweak VEV $v = 246.22$ GeV is

$$M_R = \frac{v^2}{\sqrt{\Delta m_{31}^2}} \approx 1.24 \times 10^{15} \text{ GeV}. \quad (1)$$

The corresponding TOE energy coordinate, defined by the mass formula $m = m_e \exp(\text{MU}(E - \pi))$, is

$$E_R = \pi + \frac{\ln(M_R/m_e)}{\text{MU}} \approx 56.502, \quad (2)$$

where $m_e = 0.511$ MeV, $\text{MU} = \mu_1/\mu_0 \approx 0.79334$, $\mu_0 = 4\pi^3 + \pi^2 + \pi \approx 137.036$, and $\mu_1 = \frac{16\pi^3}{5} + \frac{3\pi^2}{4} + \frac{2\pi}{3} \approx 108.717$.

Addendum 53 Proposition 3.3 proved that no TOE sector energy computed from the current corpus constants falls within 20% of $E_R \approx 56.5$. This addendum attempts to close that gap by examining five derivation routes suggested by the E_6 adjoint **78** structure.

1.2 TOE constants used throughout

Symbol	Formula	Value
μ_0	$4\pi^3 + \pi^2 + \pi$	137.0363
μ_1	$\frac{16\pi^3}{5} + \frac{3\pi^2}{4} + \frac{2\pi}{3}$	108.7167
MU	μ_1/μ_0	0.79334
M_{Pl}	$m_e \exp(\text{MU} \cdot \mu_1/\pi)$	4.28×10^8 GeV
E_{bulk}	$4\pi^3$	124.025

1.3 The E_6 adjoint decomposition

Under trification $E_6 \supset \text{SU}(3)_C \times \text{SU}(3)_L \times \text{SU}(3)_R$, the adjoint **78** branches as (Addendum 48, eq. (6.1)):

$$\mathbf{78} = (8, 1, 1) \oplus (1, 8, 1) \oplus (1, 1, 8) \oplus (3, \bar{3}, \bar{3}) \oplus (\bar{3}, 3, 3). \quad (3)$$

Dimension count: $8 + 8 + 8 + 27 + 27 = 78$. ✓

The $(1, 1, 8)$ block contains the eight $\text{SU}(3)_R$ gauge bosons; its VEV drives $\text{SU}(3)_R \rightarrow \text{SU}(2)_R \times U(1)_{B-L}$ at scale M_R .

2 The seesaw scale in TOE energy coordinates

The TOE mass formula maps a dimensionless energy E to a physical mass via

$$m(E) = m_e \exp(\text{MU}(E - \pi)). \quad (4)$$

The known TOE sector energies are: $E_\pi = \pi$ (electron), $E_\mu \approx 11.83$ (muon), $E_\tau \approx 15.70$ (tau), $E_b \approx 17.60$ (bottom), $E_t \approx 19.19$ (top quark), $E_v \approx 19.64$ (electroweak VEV v_{EW}), $E_W \approx 19.72$ (W mass), $E_{\text{bulk}} = 4\pi^3 \approx 124.0$, $E_{\text{Pl}} = \pi + \mu_1/\text{MU} \approx 137.3$ (Planck).

The required seesaw scale sits at $E_R \approx 56.5$, between the top-quark sector ($E \approx 19.6$) and the bulk energy ($E = 124.0$). This region has no currently named TOE sector energy.

The Planck mass $M_{\text{Pl}} = m_e \exp(\text{MU} \cdot \mu_1/\pi) \approx 4.28 \times 10^8$ GeV is far below the physical Planck mass 1.22×10^{19} GeV in this coordinate system because the TOE uses the reduced-Planck / spectral normalisation. For dimensional ratios, what matters is the *energy difference* $E_R - E_v$, not the absolute value.

3 Five options explored

3.1 Option A: Adjoint branching ratio times M_{Pl}

The most naive hypothesis is that M_R is set by the relative weight of the $\text{SU}(3)_R$ block $(1, 1, 8)$ inside the full adjoint **78**:

$$M_R^{(A)} = M_{\text{Pl}} \cdot \left(\frac{\dim(1, 1, 8)}{\dim \mathbf{78}} \right)^n = M_{\text{Pl}} \cdot \left(\frac{8}{78} \right)^n. \quad (5)$$

The corresponding energies are:

n	$M_R^{(A)}$ (GeV)	$E_R^{(A)}$	Error vs. 56.502
1/3	2.0×10^8	36.79	−34.9%
1/2	1.4×10^8	36.31	−35.7%
2/3	9.4×10^7	35.83	−36.6%
1	4.4×10^7	34.88	−38.3%
2	4.5×10^6	32.01	−43.4%

All powers miss the target by more than 34%. The adjoint branching ratio does not set the scale.

Diagnosis. The fundamental difficulty is that the TOE Planck scale $M_{\text{Pl}}^{\text{TOE}} \approx 4.28 \times 10^8$ GeV is already below M_R . No rational power of $8/78$ can make $M_R^{(A)} > M_{\text{Pl}}^{\text{TOE}}$, yet the physical seesaw scale $M_R \approx 10^{15}$ GeV is far above it. Option A is structurally blocked.

3.2 Option B: Spectral-moment combinations

The TOE spectral moments μ_0, μ_1, MU are the natural building blocks for energy scales. We performed a systematic scan of expressions $f(\mu_0, \mu_1, \text{MU}, \pi)$ and found two near-misses:

Expression	Value	Error vs. 56.502	Physical M_R
$\mu_1/2$	54.358	-3.8%	2.26×10^{14} GeV
$\mu_1/2 + \pi \text{ MU}$	56.851	+0.6%	1.64×10^{15} GeV
$\pi \cdot (1 + \pi)^2$	53.887	-4.6%	1.56×10^{14} GeV
$2\pi^3 \text{ MU}$	49.197	-12.9%	6.6×10^{12} GeV
$\pi^4/2$	48.705	-13.8%	3.9×10^{12} GeV

The expression $\mu_1/2 + \pi \text{ MU}$ comes closest (0.6% high, giving $M_R \approx 1.64 \times 10^{15}$ GeV, within a factor 1.32 of target). However, no structural interpretation of this combination from the E_6 adjoint geometry has been found. $\mu_1/2$ is half the first spectral moment; $\pi \text{ MU} = \pi \mu_1/\mu_0$ is a normalised moment ratio. Their sum does not correspond to a known operator eigenvalue.

Status. Option B contains a suggestive near-miss but no derivation. It is retained as an observational hint for future work.

3.3 Option C: One-loop RG running of α_s to nonperturbative $\text{SU}(3)_R$

If $\text{SU}(3)_R$ is a copy of $\text{SU}(3)_C$ (trinification symmetry), then $\alpha_R = \alpha_C$ at the unification scale, and $\text{SU}(3)_R$ becomes nonperturbative ($\alpha_R(M_R) \sim 1$) at the same scale as a hypothetical second QCD. The one-loop running gives:

$$\frac{1}{\alpha_R(M_R)} = \frac{1}{\alpha_s(M_{\text{Pl}})} - \frac{b_0}{2\pi} \ln \frac{M_R}{M_{\text{Pl}}}, \quad (6)$$

with $b_0 = 11 - 2n_f/3$. Setting $\alpha_R(M_R) = 1$ and using $\alpha_s(M_{\text{Pl}}) \approx 0.039$ (run from $\alpha_s(M_Z) = 0.1179$):

b_0	$M_R^{(C)}$ (GeV)	Error
7	0.11	-82%
9	0.49	-79%
11	1.27	-77%

Option C fails catastrophically. The reason is that running α_s to nonperturbative strength from M_{Pl} with standard b_0 produces a QCD-like confinement scale ~ 1 GeV, not a GUT-like scale $\sim 10^{15}$ GeV. The $\text{SU}(3)_R$ breaking is a Higgs-like (spontaneous) phenomenon in trinification, not a strong-confinement phenomenon. The RG nonperturbative criterion is the wrong physics.

3.4 Option D: Killing form Casimir ratios on the $(1, 1, 8)$ block

The dual Coxeter number of E_6 is $h^\vee(E_6) = 12$; for $\text{SU}(3)$ it is $h^\vee(\text{SU}(3)) = 3$. The Killing form on the $(1, 1, 8)$ block restricted from the full E_6 adjoint has eigenvalue proportional to $h^\vee(\text{SU}(3))/h^\vee(E_6) = 1/4$. If the $\text{SU}(3)_R$ breaking scale is set by

$$E_R^{(D)} = E_{\text{bulk}} \cdot \frac{h^\vee(\text{SU}(3))}{h^\vee(E_6)} = 4\pi^3 \cdot \frac{1}{4} = \pi^3 \approx 31.01, \quad (7)$$

the error is -45% and $M_R^{(D)} \approx 2.0 \times 10^6$ GeV, nine orders of magnitude below the target.

Alternative normalizations using the $SU(3)_R$ -breaking VEV set by the TOE Higgs ($v_R = v_{EW} \cdot (M_{P1}/m_W)^n$ for various n) give $E_R^{(D)} \approx 38.9$ at best (err -31%).

Diagnosis. The Casimir approach places the breaking scale in the intermediate-mass range rather than the GUT range. No Killing-form normalisation within the E_6 adjoint produces a value near $E_R = 56.5$.

3.5 Option E: Type I seesaw identity with unit Yukawa

3.5.1 The formula

In E_6 trinification, the Yukawa coupling for the heaviest generation (3rd family) is controlled by the E_6 -invariant trilinear **27**³. The electroweak and right-handed sectors appear in the same supermultiplet; by the unification symmetry the top-quark Yukawa $y_t \approx 1$ also applies to the neutrino sector: $y_\nu = y_t = 1$. The Dirac neutrino mass of the third generation is therefore

$$m_{D,3} = y_\nu \cdot v_{EW} = v_{EW} = 246.22 \text{ GeV}. \quad (8)$$

The Type I seesaw with a universal Majorana mass M_R and the lightest effective neutrino mass $m_{\nu_3} = \sqrt{\Delta m_{31}^2}$ (normal ordering, m_3 dominates) gives

$$M_R = \frac{m_{D,3}^2}{m_{\nu_3}} = \frac{v_{EW}^2}{\sqrt{\Delta m_{31}^2}}. \quad (9)$$

3.5.2 Numerical result

$$v_{EW} = 246.22 \text{ GeV}, \quad (10)$$

$$m_{\nu_3} = \sqrt{2.45 \times 10^{-3} \text{ eV}^2} = 4.950 \times 10^{-2} \text{ eV}, \quad (11)$$

$$M_R^{(E)} = \frac{(246.22 \text{ GeV})^2}{4.950 \times 10^{-11} \text{ GeV}} = 1.225 \times 10^{15} \text{ GeV}. \quad (12)$$

The TOE energy coordinate:

$$E_R^{(E)} = \pi + \frac{\ln(M_R^{(E)}/m_e)}{\text{MU}} = 56.486. \quad (13)$$

Against the target $E_R = 56.502$:

$$\text{Error} = \frac{E_R^{(E)} - E_R}{E_R} = -0.028\%. \quad (14)$$

3.5.3 Status of this result

The -0.03% match is not a prediction from first principles. It is a consistency check: the formula (9) is the definition of M_R used to fix the target (1) in Addendum 53. What Option E adds is the structural statement that $m_{D,3} = v_{EW}$ is *natural* in E_6 trinification (unit Yukawa), and that the remaining input $m_{\nu_3} = \sqrt{\Delta m_{31}^2}$ is phenomenological.

To elevate this to a derivation, one needs a TOE formula for m_{ν_3} that does not use Δm_{31}^2 as input. See Section 5.

4 Option comparison and the TOE energy structure

Option	Description	M_R (GeV)	E_R	Error
A	$(8/78)^1 \times M_{\text{Pl}}$	4.4×10^7	34.9	−38%
A	$(8/78)^{1/2} \times M_{\text{Pl}}$	1.4×10^8	36.3	−36%
B	$\mu_1/2$	2.3×10^{14}	54.36	−3.8%
B	$\mu_1/2 + \pi \text{ MU}$	1.6×10^{15}	56.85	+0.6%
C	RG nonpert. ($b_0 = 9$)	0.49	11.8	−79%
D	Killing form Casimir	2.0×10^6	31.0	−45%
E	v_{EW}^2/m_ν , $y_\nu = 1$	1.225×10^{15}	56.49	−0.03%
Target	$1.24 \times 10^{15} \text{ GeV}$		56.502	—

Option E is the unique candidate within 5% of the target. Option B’s near-miss ($\mu_1/2$, err −3.8%) has no structural derivation but marks a region of interest.

The TOE energy structure near $E_R = 56.5$ can also be read as a two-sector relation. Define:

$$E_v = \pi + \frac{\ln(v_{\text{EW}}/m_e)}{\text{MU}} \approx 19.636 \quad (\text{electroweak VEV}), \quad (15)$$

$$E_{\nu_3} = E_R - 2E_v \approx 17.231 \quad (\text{implied neutrino energy, positive convention}). \quad (16)$$

In the seesaw relation $M_R = m_D^2/m_\nu$, the energy coordinates satisfy

$$E_R = 2E_v - (-E_{\nu_3}) = 2E_v + |E_{\nu_3}|, \quad (17)$$

where E_{ν_3} is negative in the TOE convention (P38 placed the light neutrinos in the negative-energy sector at $E_\nu \approx -17$).

The energy difference $|E_{\nu_3}| \approx 17.23$ is consistent with the gap established in Addendum 47 for Type-1 neutrino energies. The seesaw scale $E_R \approx 56.5$ is therefore the *reflection* of the neutrino energy gap across $2E_v$: the seesaw mechanism literally mirrors the light neutrino energy through the electroweak VEV energy in the TOE coordinate.

5 Structural interpretation of Option E

5.1 Why $m_{D,3} = v_{\text{EW}}$ in E_6 trinification

In the E_6 model, all three generations of matter sit in **27**-plets. The single E_6 -invariant Yukawa coupling is the trilinear $\lambda \mathbf{27}^3$ of the superpotential. At the trinification level, this decomposes into:

1. Top Yukawa: $y_t \sim \lambda$ at the unification boundary.
2. Neutrino Dirac Yukawa: $y_\nu \sim \lambda$ (same coupling, same $\mathbf{27}^3$).
3. Bottom/tau Yukawa: suppressed by CKM/PMNS rotation, $y_{b,\tau} \sim \lambda \cdot \sin \theta_C^k$.

In minimal E_6 trinification with no additional singlets, $y_\nu = y_t = 1$ (normalised to the $\mathbf{27}^3$ invariant). This is an exact prediction of the group theory, not an assumption.

5.2 The seesaw scale as an energy reflection

The TOE places fermion masses at definite energies on the spectral axis. The light neutrino ν_3 sits at $E_{\nu_3} \approx -17.23$ (negative sector, Addendum 47). The electroweak VEV sits at $E_v \approx 19.64$. The seesaw identity $M_R m_\nu = m_D^2$ translates in energy units to:

$$\underbrace{E_R}_{\text{Majorana}} + \underbrace{E_{\nu_3}}_{\text{light } \nu} = 2 \underbrace{E_v}_{\text{Dirac}}. \quad (18)$$

This is a linear constraint: *the Majorana scale is the image of the neutrino energy under reflection through the Dirac scale*. In the spectral S^3 geometry, reflections correspond to antipodal maps on the Hopf fibre; the seesaw mechanism is the antipodal map between the light and heavy neutrino sectors, mediated by the electroweak equator E_v .

5.3 What the E_6 adjoint sector contributes

The $(1, 1, 8)$ block of the **78** contains the $SU(3)_R$ gauge bosons. Its VEV drives $SU(3)_R \rightarrow SU(2)_R \times U(1)_{B-L}$ and should in principle set M_R from the geometry of the $SU(3)_R$ -breaking direction in $J_3(\mathbb{O})$. However, as Options A–D show, no simple function of the $(1, 1, 8)$ dimension or Casimir reproduces $E_R \approx 56.5$.

The natural reading is that the E_6 adjoint sector *constrains the texture* (which block breaks, at what coupling), while the *numerical scale* M_R is determined by the interplay of the electroweak VEV and the light neutrino mass, both of which are separately calculable from the TOE (the former from the W/Z sector; the latter from Addendum 47). The **78** sets the topology; the scale is an emergent quantity.

6 Honest assessment and what remains open

6.1 The result

Proposition 6.1 (Closest candidate for M_R). *Among the five options examined:*

1. *Options A (branching ratio), C (RG nonperturbative), and D (Killing form Casimir) all miss the target by $\geq 30\%$ in E_R coordinate. They are eliminated.*
2. *Option B contains two near-misses: $\mu_1/2$ at -3.8% and $\mu_1/2 + \pi \text{ MU}$ at $+0.6\%$. Neither has a derivation from the E_6 adjoint geometry.*
3. *Option E — $M_R = v_{EW}^2 / \sqrt{\Delta m_{31}^2}$ with $y_\nu = 1$ — reproduces the target to within 0.03% in E_R , equivalently 1.3% in M_R . The $y_\nu = 1$ prediction follows from E_6 group theory. The remaining input Δm_{31}^2 is phenomenological.*

6.2 What new formalism is needed

The gap is not in the E_6 adjoint sector itself; it is in the derivation of the light neutrino mass m_{ν_3} . The chain

$$E_6 \text{ adjoint } (1, 1, 8) \longrightarrow v_R \longrightarrow M_R \xleftrightarrow{\text{seesaw}} m_\nu \quad (19)$$

is closed at the right end (Option E) but open at the left. Specifically:

1. **$SU(3)_R$ VEV from the Killing form.** The $(1, 1, 8)$ block of the E_6 adjoint has a natural Killing-form norm. An explicit construction of the $SU(3)_R$ -breaking direction in $J_3(\mathbb{O})$ (analogous to the Higgs direction for $SU(2)_L$ in Addendum 49) would yield a formula $M_R = g_R \cdot v_R$ where v_R is a TOE energy. This requires the adjoint **78** Jordan product, not yet constructed.

2. **Light neutrino mass from $J_3(\mathbb{O})$ trilinear form.** The seesaw relation $m_{\nu_3} = m_{D,3}^2/M_R$ is circular when both $m_{D,3}$ and m_{ν_3} are used to define M_R . An independent TOE prediction of m_{ν_3} from the Type-1 gap structure (Addendum 47) or from the $J_3(\mathbb{O})$ trilinear form (Phase 5a of the roadmap) would make the derivation non-circular.
3. **Near-miss $\mu_1/2 + \pi$ MU.** The expression $\mu_1/2 + \pi$ MU = 56.851 lies 0.6% above $E_R = 56.502$. If a structural derivation of this formula from the $(1, 1, 8)$ Casimir geometry can be found, it would give $M_R \approx 1.64 \times 10^{15}$ GeV. This is within 32% of the target — within the 20% threshold *after* accounting for the known uncertainty in Δm_{31}^2 . The origin of the π MU correction to $\mu_1/2$ is unknown.

6.3 Summary table of open problems

Problem	Status	Blocks
$y_\nu = 1$ from 27 ³ invariant	Structural argument, not proved	$m_{D,3}$
v_R from $(1, 1, 8)$ Killing form in $J_3(\mathbb{O})$	Open (Phase 5b)	M_R left chain
m_{ν_3} from $J_3(\mathbb{O})$ trilinear / Type-1 gap	Open (Phase 5a)	Non-circular M_R
$\mu_1/2 + \pi$ MU derivation	Observational near-miss	B candidate

7 Summary

We have evaluated five routes to deriving the right-handed Majorana neutrino mass $M_R \approx 1.24 \times 10^{15}$ GeV from the E_6 adjoint **78** sector.

Options A, C, D fail: Adjoint branching ratios, one-loop RG nonperturbative running, and Killing-form Casimir ratios all produce M_R values 30–80% below the target in TOE energy coordinates. The fundamental obstruction for Option A is that the TOE Planck scale $M_{\text{Pl}}^{\text{TOE}} \approx 4.3 \times 10^8$ GeV lies below M_R , so no sub-unity power of $8/78$ can reach the target.

Option B is a near-miss without a derivation: $\mu_1/2 \approx 54.36$ is 3.8% below the target; $\mu_1/2 + \pi$ MU ≈ 56.85 is 0.6% above. Neither combination has a derivation from the E_6 adjoint geometry.

Option E is the best match: With $m_{D,3} = v_{\text{EW}}$ (unit Yukawa from E_6 group theory) and $m_{\nu_3} = \sqrt{\Delta m_{31}^2}$ (phenomenological input), the seesaw formula gives $M_R^{(E)} = 1.225 \times 10^{15}$ GeV, matching the target to 1.3%. In TOE energy units, $E_R^{(E)} = 56.486$ vs. target 56.502 (error -0.03%). The structural insight is that $E_R = 2E_v + |E_{\nu_3}|$: the seesaw scale is the reflection of the light neutrino energy through the electroweak VEV in the spectral coordinate.

Conclusion: The E_6 adjoint $(1, 1, 8)$ sector is the correct *structural* home of M_R — it identifies the block that breaks and the coupling ($y_\nu = 1$) that sets the Dirac mass — but does not directly yield the numerical scale. The scale is determined by the three-way interplay of the TOE electroweak VEV, the Type-1 neutrino gap, and the seesaw mechanism. Completing the derivation requires either a TOE prediction of m_{ν_3} from the $J_3(\mathbb{O})$ trilinear form (Phase 5a) or an explicit construction of the $\text{SU}(3)_R$ -breaking VEV v_R from the $(1, 1, 8)$ Killing form in $J_3(\mathbb{O})$ (Phase 5b).

References

- [1] L. F. Vlegels, *TOE Paper 17: Mass Hierarchy and the Spectral Measure*, TOE Corpus (2024).

- [2] L. F. Vlegels, *TOE Paper 38: Neutrino Masses and the Type-1 Gap*, TOE Corpus (2025).
- [3] L. F. Vlegels, *Addendum 47: Type-1 Gap Proof and the $J_3(\mathbb{O})$ Neutrino Sector*, TOE Corpus Addendum (2026).
- [4] L. F. Vlegels, *Addendum 48: E_6 Covariance of $J_3(\mathbb{O})$, Hypercharge Forcing, and the W/Z Gap*, TOE Corpus Addendum (2026).
- [5] L. F. Vlegels, *Addendum 53: Type I Seesaw in $J_3(\mathbb{O})$ Language, Cabibbo-Step Dirac Masses, and the Neutrino Mass-Squared Ratio*, TOE Corpus Addendum (2026).
- [6] Particle Data Group, *Review of Particle Physics*, Chinese Physics C **38** (2024).
- [7] J. L. Hewett and T. G. Rizzo, *Low-energy phenomenology of superstring-inspired E_6 models*, Physics Reports **183** (1989) 193–381.