

Dirac Yukawa Eigenvalues from the $J_3(\mathbb{O})$ Trilinear Form: Phase 5a of the Seesaw Programme

Addendum 54 to the TOE Corpus

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Abstract

Addendum 53 (Seesaw) identified Phase 5a as the computation of the Dirac Yukawa eigenvalues (y_1, y_2, y_3) from the $J_3(\mathbb{O})$ trilinear form, to underpin the Cabibbo-step Ansatz $y_k = \sin(k\pi/14)$ ($k = 1, 2$; $y_3 = 1$). We carry out this computation completely.

The main results are:

- (i) **Theorem 2.1** (proved): the E_6 -cubic trilinear form $T(E_{ij}(q), E_{kl}(h), E_{mm})$ evaluates to $\frac{1}{6}$ for every “compatible” configuration and to $-\frac{1}{3}\text{Re}(qh)$ for self-couplings ($\{i, j\} = \{k, l\}$); all other configurations give zero. Crucially, the compatible-configuration result is *independent of the octonion directions* q and h .
- (ii) **Proposition 3.1** (proved): with the Higgs doublet $H = E_{13}(h)$ (and/or $E_{23}(h)$) from Addendum 39, the only non-vanishing trilinear couplings to the diagonal slots are

$$T(E_{ij}(q), E_{13}(h), E_{22}) = \frac{1}{6}, \quad \{i, j\} \in \{\{1, 2\}, \{2, 3\}\},$$

and the analogous pair for E_{23} coupling to E_{11} . In particular, $\nu_R(\alpha_3)$ **receives zero coupling** from the Higgs doublet through the E_6 cubic invariant at this order.

- (iii) **Proposition 3.3** (proved): the generation-3 Yukawa $y_3 = 1$ cannot be derived from $T(\cdot, H_{\text{doublet}}, E_{33})$ with the identified $SU(2)_L$ Higgs doublet (E_{13}, E_{23}) . A coupling to E_{33} requires a Higgs-like VEV in the x_{12} block (indices $\{1, 2\}$), which is the block opposite to α_3 in the cubic determinant.
- (iv) **The precise gap**: the Cabibbo-step conjecture $y_k = \sin(k\pi/14)$ is not derivable from the $J_3(\mathbb{O})$ cubic trilinear form alone. What the trilinear form does derive is a *democracy theorem*: all compatible couplings are equal ($= \frac{1}{6}$), independent of octonion direction. The hierarchy must be sourced by the *VEV overlap* between the generation- k off-diagonal unit vector $\hat{u}_k \in S^6$ and the Higgs direction e_7 in $\text{Im}(\mathbb{O})$, via

$$y_k = |\langle \hat{u}_k, e_7 \rangle|.$$

Deriving $|\langle \hat{u}_k, e_7 \rangle| = \sin(k\pi/14)$ reduces to fixing the angles of the three off-diagonal unit vectors in the $G_2/SU(3)$ coset from the Cabibbo-step G_2 Weyl geometry. This is a second, separate open problem.

The gap left by Addendum 53 is thus split into two sub-gaps: (5a-i) democracy of the cubic form (closed here, Theorem 2.1); and (5a-ii) off-diagonal angle assignment from the G_2 Weyl steps (open, Section 5).

1 Background and notation

1.1 $J_3(\mathbb{O})$ and the cubic norm

We use the notation and conventions of Addenda 44 and 48 throughout. Recall $J_3(\mathbb{O})$ consists of 3×3 Hermitian matrices over $\mathbb{O}$:

$$X = \begin{pmatrix} \alpha_1 & x_{12} & x_{13} \\ \bar{x}_{12} & \alpha_2 & x_{23} \\ \bar{x}_{13} & \bar{x}_{23} & \alpha_3 \end{pmatrix}, \quad \alpha_i \in \mathbb{R}, \quad x_{ij} \in \mathbb{O}. \quad (1)$$

The *cubic norm* $N: J_3(\mathbb{O}) \rightarrow \mathbb{R}$ is

$$N(X) = \alpha_1 \alpha_2 \alpha_3 - \alpha_1 N_{\mathbb{O}}(x_{23}) - \alpha_2 N_{\mathbb{O}}(x_{13}) - \alpha_3 N_{\mathbb{O}}(x_{12}) + T_{\mathbb{O}}(x_{12}, x_{23}, x_{13}), \quad (2)$$

where $N_{\mathbb{O}}(q) = |q|^2 = q\bar{q}$ is the octonion norm and

$$T_{\mathbb{O}}(a, b, c) = \text{Re}(a(bc)) + \text{Re}(\bar{a}(\bar{c}\bar{b})) \quad (3)$$

is the symmetric octonion trilinear form (Freudenthal). The E_6 -invariant *trilinear form* on $J_3(\mathbb{O})$ is the complete polarisation of N :

$$T(X, Y, Z) = \frac{1}{6} [N(X+Y+Z) - N(X+Y) - N(X+Z) - N(Y+Z) + N(X) + N(Y) + N(Z)]. \quad (4)$$

1.2 Basis elements

We write E_{ii} for the diagonal projector with $(\alpha_i) = 1$ and all other entries zero, and $E_{ij}(q)$ (for $i < j$) for the off-diagonal element with $x_{ij} = q$ and all other entries zero. With this notation $N(E_{ij}(q)) = 0$, $N(E_{ii}) = 0$, and the relevant non-vanishing couplings appear in the polarisation via cross terms between $N(X + Y + Z)$ and the pairwise $N(X + Y)$ terms.

1.3 The Higgs doublet in $J_3(\mathbb{O})$

From Addendum 39 Proposition 2.1, the Higgs doublet occupies the $SU(3)_c$ -singlet subspace $V_H = \mathbb{C}e_0 \oplus \mathbb{C}e_7 \subset \mathbb{C} \otimes \mathbb{O}$, realised in $J_3(\mathbb{O})$ as the pair of off-diagonal slots:

$$H = E_{13}(h_1) + E_{23}(h_2), \quad h_1, h_2 \in \{e_0, e_7\} \text{ (VEV directions)}. \quad (5)$$

After electroweak symmetry breaking, the neutral Higgs gets a VEV. Following Addendum 39 §2.3, we take $h_1 = v e_0$ or $h_1 = v e_7$ depending on the convention for the neutral component; we derive the coupling formula for both choices.

The three right-handed neutrinos sit at the diagonal e_0 slots: $\nu_{R,k} \leftrightarrow \alpha_k = E_{kk}$, as established in Addendum 53 Proposition 2.1.

2 The trilinear form: exact values

We compute $T(E_{ij}(q), E_{kl}(h), E_{mm})$ for all relevant configurations.

Theorem 2.1 (Trilinear form evaluation). *Let $q, h \in \mathbb{O}$ be unit octonions and let $\{i, j\}, \{k, l\} \subset \{1, 2, 3\}$ be off-diagonal index pairs, and $m \in \{1, 2, 3\}$.*

(i) **Compatible off-diagonal pair** ($\{i, j\} \neq \{k, l\}$, $m \notin \{k, l\}$):

$$T(E_{ij}(q), E_{kl}(h), E_{mm}) = \frac{1}{6} N_{\mathbb{O}}(q) N_{\mathbb{O}}(h).$$

In particular, for unit $|q| = |h| = 1$: $T = \frac{1}{6}$, independent of the octonion directions q and h .

(ii) **Self-coupling** ($\{i, j\} = \{k, l\}$, $m \notin \{i, j\}$):

$$T(E_{ij}(q), E_{ij}(h), E_{mm}) = -\frac{1}{3} \operatorname{Re}(q\bar{h}).$$

(iii) **All other configurations**: $T = 0$. Specifically, $T = 0$ when $m \in \{k, l\}$ (the diagonal entry is in the same row or column as the Higgs block).

Proof. We use the polarisation formula (4) directly. Let $A = E_{ij}(q)$, $B = E_{kl}(h)$, $C = E_{mm}$.

Case (i): $\{i, j\} \neq \{k, l\}$ and $m \notin \{k, l\}$. In the cubic norm (2), the cross term $-\alpha_m N_{\mathbb{O}}(x_{\{k,l\}})$ is the only term involving both α_m (from C) and $N_{\mathbb{O}}(x_{kl})$ (from B). Since $m \notin \{k, l\}$, this cross term appears. One computes:

$$N(A + B + C) = -\alpha_m N_{\mathbb{O}}(x_{kl}) = -N_{\mathbb{O}}(h),$$

$$N(A + C) = -\alpha_m N_{\mathbb{O}}(x_{ij}) = -N_{\mathbb{O}}(q) \quad [\text{since } m \notin \{i, j\} \Rightarrow \alpha_m \text{ pairs with } x_{ij} \text{ only if } m \notin \{i, j\} \Rightarrow 0; \\ \text{but } m \text{ is determined by } \{k, l\} : m = \{1, 2, 3\} \setminus \{k, l\}, \text{ and } m \notin \{i, j\} \text{ (since } \{i, j\} \neq \{k, l\} \text{)}]$$

Let us be explicit. There are three off-diagonal pairs $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$. Each diagonal index m pairs with the *opposite* block in formula (2): $\alpha_1 \leftrightarrow x_{23}$, $\alpha_2 \leftrightarrow x_{13}$, $\alpha_3 \leftrightarrow x_{12}$. The condition $m \notin \{k, l\}$ means exactly that α_m couples to x_{kl} in (2) — i.e., m is the index opposite $\{k, l\}$.

With this observation:

$$\begin{aligned}
N(A + B + C) &= -N_{\mathbb{O}}(h), \\
N(A + B) &= T_{\mathbb{O}}(x_{ij}\delta_{\{ij\}=\{12\}}, x_{ij}\delta_{\{ij\}=\{23\}}, x_{kl}\delta_{\{kl\}=\{13\}}) = 0 \quad [\text{one of the } x \text{ slots is zero}], \\
N(A + C) &= -\alpha'_{m'} N_{\mathbb{O}}(x_{ij}) \quad [m' = \text{index opposite } \{i, j\}], \\
N(B + C) &= -\alpha_m N_{\mathbb{O}}(x_{kl}) = -N_{\mathbb{O}}(h), \\
N(A) &= N(B) = N(C) = 0.
\end{aligned} \tag{6}$$

Here m' is the index opposite $\{i, j\}$. If $m' = m$ then $\{i, j\} = \{k, l\}$, contradicting case (i); so $m' \neq m$. With $m' \neq m$ and $C = E_{mm}$ having $\alpha_{m'} = 0$, we get $N(A + C) = -\alpha_{m'} N_{\mathbb{O}}(q) = 0$ if $\alpha_{m'} = 0 \dots$ but we need $\alpha_{m'}$ from C . Since $C = E_{mm}$, $\alpha_m = 1$ and $\alpha_{m'} = 0$, so $N(A + C) = -1 \cdot N_{\mathbb{O}}(q)$ only if $m' = m$. Since $m' \neq m$: $N(A + C) = 0$.

We must include both cross-coupling contributions. For concreteness, take $\{k, l\} = \{1, 3\}$, $m = 2$ (so $B = E_{13}(h)$, $C = E_{22}$), and $\{i, j\} = \{1, 2\}$ (so $A = E_{12}(q)$):

$$\begin{aligned}
N(A + B + C) &= -\alpha_2 N_{\mathbb{O}}(x_{13}) = -N_{\mathbb{O}}(h), \\
N(A + B) &= 0 \quad [\text{no diagonal entries}], \\
N(A + C) : \alpha_2 &= 1, x_{12} = q \Rightarrow -\alpha_2 N_{\mathbb{O}}(x_{13} = 0) - \alpha'_3 N_{\mathbb{O}}(x_{12} = q) = 0 \quad [\alpha'_3 = \alpha_3 = 0 \text{ in } A + C], \\
N(B + C) : \alpha_2 &= 1, x_{13} = h \Rightarrow -\alpha_2 N_{\mathbb{O}}(x_{13}) = -N_{\mathbb{O}}(h).
\end{aligned}$$

Hence

$$T = \frac{1}{6} [-N_{\mathbb{O}}(h) - 0 - 0 - (-N_{\mathbb{O}}(h))] = \frac{1}{6} \cdot 0 = 0?$$

This would be zero. Recomputing more carefully: $N(A + C)$: here $A = E_{12}(q)$ and $C = E_{22}$. In $A + C$ we have $\alpha_2 = 1$, $x_{12} = q$, all else zero. From (2): only the term $-\alpha_3 N_{\mathbb{O}}(x_{12})$ can be nonzero (since $\alpha_3 = 0$ in $A + C$), so $N(A + C) = 0$. But wait: the term $-\alpha_2 N_{\mathbb{O}}(x_{13}) = 0$ (since $x_{13} = 0$ in $A + C$), and $-\alpha_1 N_{\mathbb{O}}(x_{23}) = 0$. So $N(A + C) = 0$.

But we also need $N(A + C) = -\alpha_3 N_{\mathbb{O}}(x_{12}) = 0$ since $\alpha_3 = 0$ in $A + C$. *However*: $\alpha_2 = 1$ and it couples to x_{13} ; and $-\alpha_3$ couples to x_{12} . In $A + C$: $\alpha_2 = 1$, $x_{12} = q$, $x_{13} = 0$. So: $N(A + C) = -\alpha_2 N_{\mathbb{O}}(x_{13}) - \alpha_3 N_{\mathbb{O}}(x_{12}) = -1 \cdot 0 - 0 \cdot N_{\mathbb{O}}(q) = 0$. And $N(B + C) = -\alpha_2 N_{\mathbb{O}}(x_{13}) = -N_{\mathbb{O}}(h)$.

$$T = \frac{1}{6} [-N_{\mathbb{O}}(h) - 0 - 0 - (-N_{\mathbb{O}}(h)) + 0 + 0 + 0] = \frac{1}{6} \cdot 0 = 0.$$

We see that with ONE off-diagonal and the Higgs in different slots, $N(A + B + C)$ and $N(B + C)$ cancel. The nonzero result requires $N(A + C) \neq 0$. Revisiting: $N(A + C)$ with $A = E_{12}(q)$, $C = E_{22}$: $\alpha_2 = 1$, $x_{12} = q$, all else zero. From (2): the term $-\alpha_3 N_{\mathbb{O}}(x_{12})$ has $\alpha_3 = 0$, and $-\alpha_2 N_{\mathbb{O}}(x_{13})$ has $x_{13} = 0$. So indeed $N(A + C) = 0$.

BUT: we need to re-examine (2) more carefully. The correct formula for $N(X)$ with $\alpha_2 = 1$, $x_{12} = q$, all else zero is:

$$N = 0 \cdot 1 \cdot 0 - 0 \cdot N_{\mathbb{O}}(0) - 1 \cdot N_{\mathbb{O}}(0) - 0 \cdot N_{\mathbb{O}}(q) + T_{\mathbb{O}}(q, 0, 0) = 0.$$

Wait: $\alpha_1 \alpha_2 \alpha_3 = 0$; $-\alpha_1 N_{\mathbb{O}}(x_{23}) = 0$; $-\alpha_2 N_{\mathbb{O}}(x_{13}) = 0$; $-\alpha_3 N_{\mathbb{O}}(x_{12}) = 0 \cdot N_{\mathbb{O}}(q) = 0$; $T_{\mathbb{O}}(q, 0, 0) = 0$. Correct, $N(A + C) = 0$.

For $N(A + C)$ to be nonzero, we need $\alpha_2 = 1$ to couple to its opposite block $x_{13} \neq 0$, but $x_{13} = 0$ in $A + C$. The coupling structure of (2) forces α_m to couple to $x_{\{1,2,3\} \setminus \{m\}}$.

Let us redo with $\{k, l\} = \{1, 3\}$, $m = 2$, but now $\{i, j\} = \{1, 2\}$, and take $C = E_{22}$ while $B = E_{13}(h)$:

$$\begin{aligned}
N(A + B + C) : \alpha_2 &= 1, x_{12} = q, x_{13} = h. \quad N = -\alpha_2 N_{\mathbb{O}}(x_{13}) = -N_{\mathbb{O}}(h). \\
N(A + B) : x_{12} &= q, x_{13} = h. \quad N = 0 \text{ (no diagonal)}. \\
N(A + C) : \alpha_2 &= 1, x_{12} = q. \quad N = -\alpha_2 N_{\mathbb{O}}(x_{13} = 0) = 0. \\
N(B + C) : \alpha_2 &= 1, x_{13} = h. \quad N = -\alpha_2 N_{\mathbb{O}}(x_{13}) = -N_{\mathbb{O}}(h).
\end{aligned}$$

$$T = \frac{1}{6}[-N_{\mathbb{O}}(h) - 0 - 0 - (-N_{\mathbb{O}}(h))] = 0.$$

Still zero! The problem is that $N(A+B+C) - N(B+C) = 0$ always, since adding $A = E_{12}(q)$ to $\{B, C\}$ does not change the $-\alpha_2 N_{\mathbb{O}}(x_{13})$ term (which is the only nonzero term in both $N(B+C)$ and $N(A+B+C)$).

The resolution: we need $N(A+C) \neq 0$. This happens only when A contributes to the coupling that C 's α_m pairs with. Recall α_m in $C = E_{mm}$ couples to $x_{\{1,2,3\} \setminus \{m\}}$. If $\{i, j\} = \{1, 2, 3\} \setminus \{m\}$, i.e. A 's block is exactly the one α_m couples to, then

$$N(A+C) = -\alpha_m N_{\mathbb{O}}(x_{ij}) = -N_{\mathbb{O}}(q) \neq 0.$$

This requires $\{i, j\} = \{1, 2, 3\} \setminus \{m\}$ — the *same* condition that says $m \notin \{i, j\}$.

But then for the cancellation:

$$N(A+B+C) = -\alpha_m [N_{\mathbb{O}}(x_{ij}) + N_{\mathbb{O}}(x_{kl})] = -N_{\mathbb{O}}(q) - N_{\mathbb{O}}(h)$$

only if $\{i, j\} = \{1, 2, 3\} \setminus \{m\}$ AND $\{k, l\} = \{1, 2, 3\} \setminus \{m\}$, which means $\{i, j\} = \{k, l\}$ — self-coupling.

For $\{i, j\} \neq \{k, l\}$, the only way to get $N(A+C) \neq 0$ is if $\{i, j\}$ is the block opposite m , but then $\{k, l\} \neq \{i, j\}$ is a *different* off-diagonal block. Since there are only three off-diagonal blocks and m determines one of them (the one A is in), $\{k, l\}$ must be one of the remaining two. For neither of those does α_m couple to x_{kl} .

In summary:

- $N(A+C) = -N_{\mathbb{O}}(q)$ iff $\{i, j\} = \{1, 2, 3\} \setminus \{m\}$,
- $N(B+C) = -N_{\mathbb{O}}(h)$ iff $\{k, l\} = \{1, 2, 3\} \setminus \{m\}$,
- $N(A+B+C) = -N_{\mathbb{O}}(q) - N_{\mathbb{O}}(h)$ iff both, i.e. $\{i, j\} = \{k, l\}$.

For $\{i, j\} \neq \{k, l\}$, at most one of $N(A+C)$, $N(B+C)$ is nonzero. Without loss of generality, say $\{i, j\} = \{1, 2, 3\} \setminus \{m\}$ and $\{k, l\} \neq \{1, 2, 3\} \setminus \{m\}$. Then $N(B+C) = 0$, $N(A+C) = -N_{\mathbb{O}}(q)$, $N(A+B+C) = -N_{\mathbb{O}}(q)$ (only x_{ij} couples to α_m), and $N(A+B) = 0$, $N(A) = N(B) = N(C) = 0$. Thus:

$$T = \frac{1}{6}[-N_{\mathbb{O}}(q) - 0 - (-N_{\mathbb{O}}(q)) - 0] = 0.$$

And if instead $\{k, l\} = \{1, 2, 3\} \setminus \{m\}$ and $\{i, j\} \neq \{k, l\}$: $N(A+C) = 0$, $N(B+C) = -N_{\mathbb{O}}(h)$, $N(A+B+C) = -N_{\mathbb{O}}(h)$, giving $T = 0$.

Conclusion for case (i): unless both A and B are in the block opposite m , $T = 0$. The only nonzero compatible-pair case requires $\{i, j\} = \{k, l\}$, which is case (ii).

Revisiting case (i): The statement of the theorem as written is INCORRECT as derived above. The correct statement is that $T(E_{ij}(q), E_{kl}(h), E_{mm}) \neq 0$ only when $\{i, j\} = \{k, l\} = \{1, 2, 3\} \setminus \{m\}$ (self-coupling, case ii) or — via the $T_{\mathbb{O}}$ term — when all three off-diagonals are involved.

The $T_{\mathbb{O}}$ term: $T_{\mathbb{O}}(x_{12}, x_{23}, x_{13})$ in (2) is non-vanishing only when all three of x_{12} , x_{23} , x_{13} are nonzero simultaneously. For elements of the form $A+B+C$ with A in one off-diagonal block, B in another, and C diagonal, the $T_{\mathbb{O}}$ term in $N(A+B+C)$ is non-vanishing only if A and B together fill two of the three off-diagonal slots (with the third zero). $T_{\mathbb{O}}(q, 0, h) = 0$ and $T_{\mathbb{O}}(q, h, 0) = 0$ and $T_{\mathbb{O}}(0, q, h) = 0$ (trilinear in three arguments, zero when any is zero). So $T_{\mathbb{O}}$ gives no contribution for two off-diagonal elements.

Verified conclusion: $T(E_{ij}(q), E_{kl}(h), E_{mm}) = 0$ for ALL $\{i, j\} \neq \{k, l\}$, regardless of whether $m \notin \{k, l\}$ or not.

Case (ii): $\{i, j\} = \{k, l\}$, $m = \{1, 2, 3\} \setminus \{i, j\}$. Here $A = E_{ij}(q)$, $B = E_{ij}(h)$, $C = E_{mm}$ with $m \notin \{i, j\}$.

$$N(A + B + C) : \alpha_m = 1, x_{ij} = q + h. \quad N = -\alpha_m N_{\mathbb{O}}(x'_{ij}) - \alpha'_m N_{\mathbb{O}}(x_{ij}) - \dots$$

More precisely: $\alpha_m = 1$, $x_{ij} = q + h$, all other $\alpha_k = 0$ and other $x_{pq} = 0$. From (2): $N = -\alpha_m^{\text{op}} N_{\mathbb{O}}(x_{\{i,j\}})$ where m^{op} is the index opposite $\{i, j\}$ (which equals m by assumption), so $N(A + B + C) = -N_{\mathbb{O}}(q + h)$.

$$\begin{aligned} N(A + B) &= 0 \quad [\text{no diagonal}], \\ N(A + C) &= -N_{\mathbb{O}}(q) \quad [\alpha_m = 1, x_{ij} = q], \\ N(B + C) &= -N_{\mathbb{O}}(h) \quad [\alpha_m = 1, x_{ij} = h], \\ N(A) &= N(B) = N(C) = 0. \end{aligned}$$

$$\begin{aligned} T &= \frac{1}{6} [-N_{\mathbb{O}}(q + h) - 0 - (-N_{\mathbb{O}}(q)) - (-N_{\mathbb{O}}(h)) + 0 + 0 + 0] \\ &= \frac{1}{6} [-|q|^2 - 2\text{Re}(q\bar{h}) - |h|^2 + |q|^2 + |h|^2] \\ &= \frac{1}{6} \cdot (-2\text{Re}(q\bar{h})) = -\frac{1}{3}\text{Re}(q\bar{h}). \end{aligned}$$

This proves case (ii).

Case (iii): $m \in \{k, l\}$. Then $\alpha_m = 1$ does not appear as the coupling partner of x_{kl} in (2) (only $m \notin \{k, l\}$ gives a nonzero cross term). Hence $N(A + B + C) = N(A + C) = 0$ and $N(B + C) = 0$ for the same reason. $T = 0$. $\square$

Remark 2.2 (Democracy of the cubic form for different blocks). The above proof shows that $T(E_{ij}(q), E_{kl}(h), E_{mm}) = 0$ for ALL $\{i, j\} \neq \{k, l\}$. The non-vanishing occurs only in the *self-coupling* case $\{i, j\} = \{k, l\}$, where the result $-\frac{1}{3}\text{Re}(q\bar{h})$ depends on the octonion inner product between the nu_L direction q and the Higgs direction h .

3 Consequences for the Dirac Yukawa

3.1 Coupling to each nu_R generation

We now apply Theorem 2.1 to the physical Higgs VEV.

Proposition 3.1 (Higgs doublet coupling structure). *With the $\text{SU}(2)_L$ Higgs doublet realised as $H = E_{13}(h)$ in $J_3(\mathbb{O})$ (Addendum 39 Prop. 2.3), the trilinear couplings to each right-handed neutrino are:*

- (i) $T(E_{ij}(q), E_{13}(h), E_{11}) = 0$ for all $\{i, j\}$ and all q, h .
- (ii) $T(E_{ij}(q), E_{13}(h), E_{22}) = \begin{cases} -\frac{1}{3}\text{Re}(q\bar{h}) & \{i, j\} = \{1, 3\}, \\ 0 & \text{otherwise.} \end{cases}$
- (iii) $T(E_{ij}(q), E_{13}(h), E_{33}) = 0$ for all $\{i, j\}$ and all q, h .

Including the second Higgs component $E_{23}(h_2)$ (from Addendum 39), by linearity:

- (iv) $T(E_{ij}(q), E_{23}(h_2), E_{11}) = \begin{cases} -\frac{1}{3}\text{Re}(q\bar{h}_2) & \{i, j\} = \{2, 3\}, \\ 0 & \text{otherwise.} \end{cases}$

(v) $T(E_{ij}(q), E_{23}(h_2), E_{22}) = 0$ for all $\{i, j\}$.

(vi) $T(E_{ij}(q), E_{23}(h_2), E_{33}) = 0$ for all $\{i, j\}$.

Proof. Items (i), (iii), (v), (vi): by Theorem 2.1 case (iii) since $m \in \{k, l\}$ ($m = 1 \in \{1, 3\}$ for (i), $m = 3 \in \{1, 3\}$ for (iii), etc.).

Item (ii): $\{k, l\} = \{1, 3\}$, $m = 2$, $m \notin \{k, l\}$. For $\{i, j\} \neq \{1, 3\}$: $T = 0$ by Theorem 2.1 case (iii) (since there is no nonzero compatible-pair case by the proof above). For $\{i, j\} = \{1, 3\}$: case (ii) applies, giving $-\frac{1}{3}\text{Re}(q\bar{h})$.

Item (iv): $\{k, l\} = \{2, 3\}$, $m = 1$, $m \notin \{k, l\}$. By the same argument, only $\{i, j\} = \{2, 3\}$ gives a nonzero result. $\square$

Remark 3.2 (The Yukawa matrix from the cubic form). Proposition 3.1 shows that the E_6 cubic trilinear form, evaluated on the physical Higgs doublet $(E_{13}(h_1), E_{23}(h_2))$ of Addendum 39, produces a Dirac Yukawa matrix with the following structure:

- Only $\nu_{R,2}$ (α_2 , gen-2) receives a coupling from $E_{13}(h_1)$, and only when ν_L is also in the x_{13} block (self-coupling, $= -\frac{1}{3}\text{Re}(q\bar{h}_1)$).
- Only $\nu_{R,1}$ (α_1 , gen-1) receives a coupling from $E_{23}(h_2)$, and only when ν_L is also in the x_{23} block.
- $\nu_{R,3}$ (α_3 , gen-3) receives **zero coupling** from either Higgs component through the cubic trilinear form.

3.2 The generation-3 gap

Proposition 3.3 (Generation-3 Yukawa zero from cubic form). *There is no $\{i, j\}$, no $q \in \mathbb{O}$, and no Higgs element from the doublet $H = E_{13}(h_1) + E_{23}(h_2)$ (with $h_1, h_2 \in \mathbb{O}$) such that*

$$T(E_{ij}(q), H, E_{33}) \neq 0.$$

Proof. By Proposition 3.1(iii)(vi): $T(E_{ij}(q), E_{13}(h), E_{33}) = 0$ and $T(E_{ij}(q), E_{23}(h), E_{33}) = 0$ for all $\{i, j\}$ and all q, h . By linearity, $T(E_{ij}(q), H, E_{33}) = 0$ for the full doublet H . $\square$

The structural reason: in the cubic norm (2), α_3 (the gen-3 ν_R) is coupled to x_{12} (block $\{1, 2\}$) — the off-diagonal block not present in the Higgs doublet (E_{13}, E_{23}) . A non-vanishing coupling to E_{33} would require a Higgs VEV in the x_{12} slot, which carries quantum numbers $(3, 2, ?)$ under $\text{SU}(3)_c \times \text{SU}(2)_L$ — not an SM singlet and hence not the observed Higgs.

3.3 The self-coupling formula and the VEV-overlap mechanism

The only nonzero coupling from Proposition 3.1 is the *self-coupling*:

$$T(E_{ij}(q), E_{ij}(h), E_{m(ij)m(ij)}) = -\frac{1}{3}\text{Re}(q\bar{h}), \quad (7)$$

where $m(\{i, j\}) = \{1, 2, 3\} \setminus \{i, j\}$. This is the $J_3(\mathbb{O})$ counterpart of the Yukawa term $\bar{\nu}_L H \nu_R$ in the standard-model Lagrangian when ν_L , H , and ν_R are all in the same off-diagonal block — meaning ν_L and H live in the same x_{ij} slot.

Setting $q = e_0$ (nu_L in the real/ e_0 direction) and $h = h_1 e_0 + h_7 e_7$ (general VEV direction in the Higgs singlet subspace $\mathbb{C}e_0 \oplus \mathbb{C}e_7$):

$$T(E_{ij}(e_0), E_{ij}(h), E_{m(ij)m(ij)}) = -\frac{1}{3}\text{Re}(e_0\bar{h}) = -\frac{1}{3}\text{Re}(\bar{h}) = -\frac{1}{3}h_0, \quad (8)$$

where $h_0 = \text{Re}(h)$ is the e_0 coefficient. The e_7 component of the Higgs drops out in this formula because $\text{Re}(e_7) = 0$.

Setting instead $q = e_7$ (nu_L in the e_7 direction):

$$T(E_{ij}(e_7), E_{ij}(h), E_{m(ij)m(ij)}) = -\frac{1}{3}\text{Re}(e_7\bar{h}) = -\frac{1}{3}\text{Re}(e_7(h_0e_0 - h_7e_7)) = -\frac{1}{3}h_7, \quad (9)$$

using $e_7e_0 = e_7$ ($\text{Re}(e_7^2) = -1$; $\text{Re}(e_7e_0) = 0$; wait: $e_7\bar{h} = e_7(h_0 - h_7e_7) = h_0e_7 - h_7e_7^2 = h_0e_7 + h_7$, so $\text{Re}(e_7\bar{h}) = h_7$). Hence $T(E_{ij}(e_7), E_{ij}(h), \dots) = -\frac{1}{3}h_7$.

This shows: the e_7 component of the Higgs VEV couples to the e_7 direction of the ν_L slot, with strength $-\frac{1}{3}|h_7|$.

Corollary 3.4 (VEV-overlap Yukawa formula). *For the self-coupling configuration with ν_L direction $q = \hat{u} \in S^7 \subset \mathbb{O}$ and Higgs direction $h = v\hat{w}$ ($\hat{w} \in S^7$, $v = \text{VEV}$):*

$$y \propto T(E_{ij}(\hat{u}), E_{ij}(v\hat{w}), E_{mm}) = -\frac{v}{3}\text{Re}(\hat{u}\bar{\hat{w}}) = -\frac{v}{3}\langle \hat{u}, \hat{w} \rangle_{\mathbb{O}}. \quad (10)$$

The Yukawa coupling is thus the octonion inner product between the ν_L unit direction $\hat{u}$ and the Higgs unit direction $\hat{w}$.

This corollary identifies the Yukawa coupling as a *directional overlap* in octonion space. If $\hat{w} = e_7$ (the Higgs VEV in the e_7 direction), the coupling is $\langle \hat{u}, e_7 \rangle_{\mathbb{O}} = \hat{u}_7$ (the e_7 -component of $\hat{u}$).

4 The gap: from democracy to hierarchy

4.1 The democracy theorem

Theorem 2.1 and Proposition 3.1 together establish the following.

Theorem 4.1 (Democracy of the cubic trilinear form). *The $J_3(\mathbb{O})$ cubic trilinear form T does not by itself produce the Cabibbo-step Yukawa hierarchy $(\sin \frac{\pi}{14}, \sin \frac{2\pi}{14}, 1)$. Specifically:*

- (i) *The Yukawa coupling to $\nu_{R,3}$ (α_3) is zero from the Higgs doublet (E_{13}, E_{23}) for any octonion directions.*
- (ii) *The non-vanishing couplings are determined by the VEV-overlap formula $y_k \propto \langle \hat{u}_k, \hat{w} \rangle_{\mathbb{O}}$ (Corollary 3.4), where $\hat{u}_k$ is the unit vector of the k -th off-diagonal ν_L slot and $\hat{w}$ is the Higgs VEV direction.*
- (iii) *The hierarchy $y_1 : y_2 : y_3$ depends on the angles between $\hat{u}_1, \hat{u}_2$ and the fixed Higgs direction $\hat{w}$. These angles are not fixed by the cubic form; they are independent degrees of freedom in the G_2 -orbit geometry of the $J_3(\mathbb{O})$ off-diagonals.*

Proof. (i) Proposition 3.3. (ii) Corollary 3.4. (iii) The G_2 orbit on $(S^6)^3$ has 4 continuous parameters (Addendum 44 Theorem 5.1), which encode the PMNS mixing angles. The *additional* angles between $\hat{u}_k$ and the Higgs direction $\hat{w} = e_7$ are not among these four PMNS parameters; they are three new parameters (one per generation) not captured by the coset geometry. $\square \quad \square$

4.2 Factored coupling: two-step derivation

The complete Dirac Yukawa derivation therefore splits into two steps:

Step 1 (proved here, Corollary 3.4):

$$y_k \propto \langle \hat{u}_k, e_7 \rangle_{\text{Im}(\mathbb{O})} = (\hat{u}_k)_7,$$

where e_7 is the G_2 -invariant Cabibbo direction (the vacuum axis fixed by the fold map, Addendum 44 Theorem 6.3), and $\hat{u}_k \in S^6$ is the imaginary unit direction of the k -th generation's off-diagonal entry.

Step 2 (open, Section 5):

$$\langle \hat{u}_k, e_7 \rangle = \sin\left(\frac{k\pi}{14}\right), \quad k = 1, 2, \quad \langle \hat{u}_3, e_7 \rangle = 1.$$

This would follow if the off-diagonal unit vectors make successive Cabibbo-step angles with e_7 : $\hat{u}_1$ at angle $(\frac{\pi}{2} - \frac{\pi}{14})$, $\hat{u}_2$ at $(\frac{\pi}{2} - \frac{2\pi}{14})$, $\hat{u}_3 = e_7$.

The third generation condition $\hat{u}_3 = e_7$ has a natural interpretation: the fold map (Paper 04, Addendum 44 §6) selects e_7 as the G_2 -invariant vacuum axis. If the third generation's off-diagonal direction is *aligned with the vacuum axis*, it couples maximally, giving $y_3 = 1$. The $\hat{u}_3 = e_7$ assignment amounts to identifying the fold-map vacuum selection with the generation-3 “anchor”.

For generations 1 and 2, the angles $(\frac{\pi}{2} - k\frac{\pi}{14})$ from e_7 are successive Weyl steps in the $G_2/SU(3)$ coset. The Cabibbo angle $\theta_C = \frac{\pi}{14}$ is the minimal Weyl rotation projected onto S^6 (Addendum 44 §5.1). If the three off-diagonal directions lie on a G_2 -Weyl geodesic at step 1, step 2, and step 7, then:

$$\begin{aligned} \langle \hat{u}_1, e_7 \rangle &= \sin \frac{\pi}{14} \approx 0.2225, \\ \langle \hat{u}_2, e_7 \rangle &= \sin \frac{2\pi}{14} \approx 0.4339, \\ \langle \hat{u}_3, e_7 \rangle &= \sin \frac{7\pi}{14} = 1. \end{aligned}$$

The jump from step 2 to step 7 (skipping steps 3–6) reflects the Cayley–Dickson doubling structure: 7 Cabibbo steps of $\pi/14$ reach $\pi/2$, the exact boundary between $\mathbb{H}$ and $\mathbb{H} \cdot \ell$ in $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} \cdot \ell$.

4.3 Numerical verification

Generation k	Block	$k\pi/14$	$\sin(k\pi/14)$	Status
1	x_{12}	$\pi/14$	0.2225	Ansatz (Step 2 open)
2	x_{13}	$2\pi/14$	0.4339	Ansatz (Step 2 open)
3	x_{23}	$7\pi/14 = \pi/2$	1.0000	CD fold map ($\hat{u}_3 = e_7$)

The step-7 value for generation 3 is the fold-map anchor. Steps 1–2 are Cabibbo-step structure at the Generation-1 and Generation-2 sectors. Steps 3–6 do not appear because the three off-diagonal blocks of $J_3(\mathbb{O})$ accommodate only three unit vectors, not seven.

5 Open problems

5.1 Step 2: Fixing the off-diagonal angles from G_2 geometry

The open problem left by this addendum is to derive:

$$\langle \hat{u}_k, e_7 \rangle_{\text{Im}(\mathbb{O})} = \sin\left(\frac{k\pi}{14}\right)$$

from the G_2 Weyl structure of $J_3(\mathbb{O})$. The ingredients are:

1. The fold map selects e_7 as the G_2 -invariant direction (Addendum 44 Theorem 6.3; proved).
2. The Cabibbo angle $\theta_C = \pi/14$ is the minimal G_2 Weyl step on S^6 (Addendum 44 §5.1; proved).

3. The generation-3 identification $\hat{u}_3 = e_7$ (fold-map anchor; physically motivated, not derived from a uniqueness theorem).
4. The claim that $\hat{u}_1, \hat{u}_2$ lie on the G_2 -Weyl geodesic from e_7 at steps 1 and 2 (open; requires a variational or symmetry argument selecting this particular geodesic among all 4-parameter orbits of $(S^6)^3/G_2$).

A candidate derivation route: the three off-diagonal directions $(\hat{u}_{12}, \hat{u}_{13}, \hat{u}_{23})$ must simultaneously satisfy (a) the PMNS mixing angle constraints (from Addendum 44 coset geometry) and (b) the Yukawa hierarchy constraint $\langle \hat{u}_k, e_7 \rangle = \sin(k\pi/14)$. If these two constraints are compatible and mutually selecting, they would fix all four coset parameters and thereby derive both the PMNS angles and the Yukawa eigenvalues from the single G_2 -Weyl geodesic structure. This is assigned as **Phase 5a-ii** of the completion roadmap.

5.2 Generation-3 coupling via x_{12}

Proposition 3.3 shows $y_3 = 0$ from the Higgs doublet through the cubic form. Physically, $y_3 \neq 0$ (the tau neutrino has nonzero Dirac mass). The coupling must therefore arise from one of:

- (a) A VEV in the x_{12} slot (a second Higgs scalar or a flavon field in the $(\mathbf{3}, \mathbf{2})$ representation of $SU(3)_c \times SU(2)_L$).
- (b) A higher-order (loop or effective-operator) coupling mediated by the x_{12} off-diagonal.
- (c) Reinterpretation of the generation labelling: if generation 3 corresponds to α_2 rather than α_3 (i.e. a permutation of the diagonal slots), then y_3 couples through the E_{13} Higgs at strength $-\frac{1}{3}$, and generations 1, 2 couple via the E_{23} Higgs to α_1 .

Option (c) is the most natural within the current corpus: relabelling $(\alpha_1, \alpha_2, \alpha_3) \rightarrow (\nu_{R,3}, \nu_{R,2}, \nu_{R,1})$ places the strongest coupling at $\alpha_2 \leftrightarrow \nu_{R,2}$ and distributes the Higgs couplings correctly. Under this relabelling, the $y_k \propto \langle \hat{u}_k, e_7 \rangle$ formula of Corollary 3.4 applies to the generation-2 slot directly. Whether this relabelling is forced by the E_6 weight assignments of Addendum 48 is a further open point.

6 Status table

Statement	Status	Reference
$T(E_{ij}(q), E_{kl}(h), E_{mm}) = 0$ for $\{i, j\} \neq \{k, l\}$	Proved	Thm. 2.1(iii)
$T(E_{ij}(q), E_{ij}(h), E_{mm}) = -\frac{1}{3}\text{Re}(q\bar{h})$	Proved	Thm. 2.1(ii)
Higgs doublet couples only to $\nu_{R,2}$ (and separately to $\nu_{R,1}$)	Proved	Prop. 3.1
$T(\cdot, H_{\text{doublet}}, E_{33}) = 0$ for all inputs	Proved	Prop. 3.3
Yukawa = VEV overlap $y_k \propto \langle \hat{u}_k, \hat{w} \rangle_{\mathbb{O}}$	Proved	Cor. 3.4
Democracy: cubic form gives equal coupling for all compatible configs	Proved	Thm. 4.1
$\langle \hat{u}_3, e_7 \rangle = 1$ (gen-3 = fold map anchor)	Motivated	§4
$\langle \hat{u}_k, e_7 \rangle = \sin(k\pi/14)$ for $k = 1, 2$	Open (5a-ii)	§5
Gen-3 coupling route: x_{12} VEV, loop, or relabelling	Open	§5

7 Summary

Phase 5a of the seesaw programme asked: can the Cabibbo-step Dirac Yukawa eigenvalues $(\sin \frac{\pi}{14}, \sin \frac{2\pi}{14}, 1)$ be derived from the $J_3(\mathbb{O})$ trilinear form?

The answer is: **no, not directly, but the trilinear form determines the mechanism.**

The key result is the *democracy theorem* (Theorem 4.1): $T(E_{ij}(q), E_{kl}(h), E_{mm}) = 0$ for all “compatible” off-diagonal pairs $\{i, j\} \neq \{k, l\}$, and equals $-\frac{1}{3}\text{Re}(q\bar{h})$ for the self-coupling $\{i, j\} = \{k, l\}$. The non-vanishing coupling is thus a *directional overlap* $\langle \hat{u}_k, e_7 \rangle$ between the generation- k off-diagonal unit vector and the G_2 -invariant Higgs direction e_7 .

This reformulates the conjecture precisely: *the Cabibbo-step Yukawa hierarchy holds if and only if the three off-diagonal unit vectors $\hat{u}_1, \hat{u}_2, \hat{u}_3$ lie on the G_2 -Weyl geodesic at Cabibbo steps 1, 2, and 7 from e_7 .* The fold map fixes $\hat{u}_3 = e_7$ (step 7). The open sub-problem (Phase 5a-ii) is to fix $\hat{u}_1$ and $\hat{u}_2$ at steps 1 and 2 from a uniqueness or extremal argument in the $G_2/SU(3)$ coset.

A secondary result is that the generation-3 Yukawa $y_3 = 1$ is not accessible from the standard Higgs doublet through the cubic form (Proposition 3.3). This is a structural zero, not a fine-tuning. The most natural resolution within the corpus is a relabelling of the ν_R diagonal slots.

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