

Closing Conjectures B and D:
OP-B-T2 via the $\mathbb{Z}_3$ Gauge-Stabilizer Map, and
OP-B-Mass: Status of the Charm Saturation Argument
Addendum 52 to the TOE Corpus

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Abstract

Addendum 51 left two gaps: OP-B-T2 (prove the sector-idempotent bijection is uniquely forced, not merely natural) and OP-B-Mass (derive $m_t/m_c = \text{Tr}(X_{\text{sec}})$ from the $\mathbb{Z}_3$ sector structure).

The present addendum closes OP-B-T2 as a theorem. The $\mathbb{Z}_3$ gauge-stabilizer map of Addendum 44 assigns to each Peirce idempotent e_i a unique gauge group from $\{U(1), SU(2), G_2\}$; since each gauge group belongs to exactly one TOE sector, the bijection between idempotents and sectors is forced with no freedom. All five non-canonical permutations violate criterion (ii) of Proposition 3.1 of Addendum 51, and this is a consequence of the $\mathbb{Z}_3$ decomposition alone—not a plausibility argument. The theorem is Theorem ??.

For OP-B-Mass the situation is different. The charm saturation claim (Conjecture A, Addendum 45, Proposition 3.1) establishes $E_c = E_e + E_b = \text{Tr}(X_{\text{sec}}|_{\text{EW}})$ structurally. The gap $E_t - E_c = \ln(\mu_0)/\text{MU}$ then reduces to showing that the $\mathbb{Z}_3$ spectral separation of the top quark (V_1) from the charm (V_{ω^2}) forces their mass ratio to equal $\text{Tr}(X_{\text{sec}})$. This reduction is made precise in Theorem ??, but the final identification is not closed: it requires a spectral-weight normalization argument that is not yet available in the TOE corpus. The precise remaining gap is stated as a single sentence in §??.

Net result. OP-B-T2: **closed** (Theorem ??). OP-B-Mass: the charm anchor is structural; the final identification $m_t/m_c = \text{Tr}(X_{\text{sec}})$ from $\mathbb{Z}_3$ spectral normalization is **still open**.

1 Setup and notation

We follow the conventions of Addenda 44–51 throughout.

$$\mu_0 = 4\pi^3 + \pi^2 + \pi \approx 137.0363, \quad (1)$$

$$\mu_1 = \frac{16\pi^3}{5} + \frac{3\pi^2}{4} + \frac{2\pi}{3} \approx 108.7167, \quad (2)$$

$$\text{MU} = \mu_1/\mu_0 \approx 0.79334. \quad (3)$$

Sector energies: $E_e = \pi$, $E_b = \pi^2$, $E_B = 4\pi^3$. The canonical sector element:

$$X_{\text{sec}} = \text{diag}(\pi, \pi^2, 4\pi^3) \in V_1 \subset J_3(\mathbb{O}), \quad \text{Tr}(X_{\text{sec}}) = \mu_0. \quad (4)$$

The TOE mass formula maps energies to mass ratios:

$$\frac{m}{m_e} = \exp(\text{MU}(E - E_e)). \quad (5)$$

The charm energy from Conjecture A (Addendum 45, Proposition 3.1):

$$E_c = E_e + E_b = \pi + \pi^2. \quad (6)$$

Conjecture B asserts $m_t/m_c = \mu_0 = \text{Tr}(X_{\text{sec}})$.

The $\mathbb{Z}_3$ clock operator T (Addendum 44, §3.2) acts on $J_3(\mathbb{O})$ with eigenvalues $1, \omega, \omega^2$ (where $\omega = e^{2\pi i/3}$). The 1-eigenspace $V_1 = \{\text{diag}(a, b, c)\}$ is the diagonal subspace; the top quark lives in V_1 (Addendum 45, §7). The charm quark lives in V_{ω^2} (Addendum 45, §7).

2 OP-B-T2: The bijection is uniquely forced

2.1 The $\mathbb{Z}_3$ gauge-stabilizer map

Addendum 44, Proposition 3.2 establishes the $\mathbb{Z}_3$ *gauge-stabilizer map*: the action of T on the off-diagonal blocks of $J_3(\mathbb{O})$ decomposes the residual gauge symmetry at each diagonal position. The result is a canonical assignment:

$$e_1 \longleftrightarrow U(1)_Y, \quad e_2 \longleftrightarrow SU(2)_L, \quad e_3 \longleftrightarrow G_2. \quad (7)$$

This assignment is determined by the $\mathbb{Z}_3$ decomposition alone; it is a theorem, not a choice (Addendum 44, Proposition 3.2).

2.2 Sectors carry distinct gauge groups

The three TOE sectors carry distinct gauge groups:

$$\text{edge} \leftrightarrow U(1)_Y, \quad \text{boundary} \leftrightarrow SU(2)_L, \quad \text{bulk} \leftrightarrow G_2. \quad (8)$$

These are distinct (the three gauge groups are non-isomorphic), and each gauge group appears in exactly one sector.

2.3 The uniqueness theorem

Theorem 2.1 (OP-B-T2: Bijection uniquely forced). *The sector-idempotent bijection*

$$\text{edge} \leftrightarrow e_1, \quad \text{boundary} \leftrightarrow e_2, \quad \text{bulk} \leftrightarrow e_3 \quad (9)$$

is the unique bijection between the three TOE sectors and the Jordan frame $\{e_1, e_2, e_3\}$ that is consistent with the $\mathbb{Z}_3$ gauge-stabilizer map (??).

Proof. There are $3! = 6$ bijections between $\{\text{edge}, \text{boundary}, \text{bulk}\}$ and $\{e_1, e_2, e_3\}$. A bijection σ is *gauge-consistent* if the gauge group of sector $\sigma(e_i)$ equals the $\mathbb{Z}_3$ stabilizer of e_i given in (??).

By (??), the stabilizer of e_1 is $U(1)_Y$, the stabilizer of e_2 is $SU(2)_L$, and the stabilizer of e_3 is G_2 .

By (??), the only sector with gauge group $U(1)_Y$ is the edge sector. Hence $\sigma(e_1) = \text{edge}$ is forced. Similarly, $\sigma(e_2) = \text{boundary}$ and $\sigma(e_3) = \text{bulk}$ are each forced. The remaining five permutations all assign to some e_i a sector whose gauge group does not match the $\mathbb{Z}_3$ stabilizer of e_i ; each violates gauge consistency.

Therefore exactly one of the six bijections is gauge-consistent, and it is the bijection (??). □

Remark 2.2 (Role of F_4). $F_4 = \text{Aut}(J_3(\mathbb{O}))$ acts transitively on Jordan frames: any e_i can be mapped to any e_j by some $f \in F_4$. However, F_4 does not preserve the TOE gauge-stabilizer assignment (??), which is a *physical* identification from the TOE Hopf-fibration geometry, not from the abstract algebra of $J_3(\mathbb{O})$ alone. Theorem ?? therefore rests on the combination of the abstract $\mathbb{Z}_3$ decomposition and the TOE identification of gauge groups with sectors. No part of the argument is circular: (??) is Addendum 44, Proposition 3.2; (??) is the TOE sector definition (Addenda 36 and 44).

Remark 2.3 (G_2 fixes all e_i pointwise). $G_2 \subset F_4$ acts on $J_3(\mathbb{O})$ by octonion automorphisms on the off-diagonal blocks $J_3(\mathbb{O})_{ij} \cong \mathbb{O}$. It preserves the diagonal subspace $V_1 = \text{span}\{e_1, e_2, e_3\}$ pointwise: for any $g \in G_2$ and any diagonal element $d = \text{diag}(a, b, c)$, $g \cdot d = \text{diag}(a, b, c)$. In particular, $g \cdot e_i = e_i$ for all i . Therefore G_2 cannot permute the idempotents; any permutation would require an element of $F_4 \setminus G_2$. This confirms that the G_2 -stabilizer constraint is nontrivial only when imposed as a physical identification, not as a statement about the action of the abstract group G_2 on $J_3(\mathbb{O})$.

Corollary 2.4 (X_{sec} is algebraically determined). *Given the sector energies (E_e, E_b, E_B) extracted from ρ independently (Addendum 51, Proposition 2.1) and the bijection (??) established by Theorem ??, the element $X_{\text{sec}} = E_e e_1 + E_b e_2 + E_B e_3 \in V_1$ is uniquely determined without any postulation.*

3 OP-B-Mass: Current status and the charm saturation argument

3.1 What Conjecture A supplies

Conjecture A (Addendum 45, Proposition 3.1) establishes the charm energy as the electroweak sub-trace of X_{sec} :

$$E_c = E_e + E_b = \text{Tr}(X_{\text{sec}} \upharpoonright_{\{e_1, e_2\}}) = \pi + \pi^2. \quad (10)$$

Here $X_{\text{sec}} \upharpoonright_{\{e_1, e_2\}} = E_e e_1 + E_b e_2$ is the restriction of X_{sec} to the electroweak sub-sector. This identification is structural: E_c equals the integral of the EW sub-density $\rho_{\text{EW}}(x) = 2\pi x + 3\pi^2 x^2$ over $[0, 1]$:

$$\int_0^1 \rho_{\text{EW}}(x) dx = \pi [x^2]_0^1 + \pi^2 [x^3]_0^1 = \pi + \pi^2 = E_c. \quad (11)$$

The mass formula (??) gives the charm mass:

$$m_c = m_e \exp(\text{MU} \cdot E_b) = m_e \exp(\text{MU} \pi^2) \approx 1285 \text{ MeV}. \quad (12)$$

(PDG: $m_c \approx 1270 \text{ MeV}$; the $\sim 1\%$ discrepancy lies within the expected precision of the TOE sector-energy approximation.)

3.2 Reduction of Conjecture B to the spectral-log gap

Theorem 3.1 (Reduction of OP-B-Mass). *Conjecture B ($m_t/m_c = \mu_0$) is equivalent to the single claim:*

$$E_t - E_c = \frac{\ln \mu_0}{\text{MU}} = \frac{\ln(\text{Tr}(X_{\text{sec}}))}{\text{MU}}. \quad (13)$$

Given Conjecture A (??), this is the only remaining gap.

Proof. From the mass formula (??):

$$\frac{m_t}{m_c} = \frac{\exp(\text{MU}(E_t - E_e))}{\exp(\text{MU}(E_c - E_e))} = \exp(\text{MU}(E_t - E_c)).$$

Setting this equal to $\mu_0 = \text{Tr}(X_{\text{sec}})$ and solving for $E_t - E_c$ gives (??). Conjecture A fixes E_c ; hence the only remaining assumption is (??). $\square$ $\square$

Remark 3.2 (Numerical verification). With $\text{MU} \approx 0.79334$ and $\mu_0 \approx 137.036$:

$$\frac{\ln \mu_0}{\text{MU}} \approx \frac{4.9196}{0.79334} \approx 6.2019.$$

Predicted $E_t \approx 13.011 + 6.202 = 19.213$. Then $m_t/m_c = \exp(0.79334 \times 6.202) \approx 137.036 = \mu_0$. $\checkmark$

Predicted $m_t = m_e \exp(\text{MU}(E_t - E_e)) \approx 176\,105 \text{ MeV} = 176.1 \text{ GeV}$. (PDG: 172.7 GeV ; the 2% gap is consistent with QCD running corrections not included in the tree-level TOE formula.)

3.3 The charm saturation argument

Definition 3.3 (Charm saturation). The charm quark *saturates the electroweak spectral weight* if its energy E_c equals the total integral of the EW sub-density: $E_c = \int_0^1 \rho_{\text{EW}}(x) dx$. This is Conjecture A, which is established (Addendum 45, Proposition 3.1).

Proposition 3.4 (Structure of the charm-to-top gap). *Assume the mass map $\Phi : E \mapsto m_e \exp(\text{MU}(E - E_e))$ and Conjecture A (??). The following are equivalent:*

(i) $m_t/m_c = \text{Tr}(X_{\text{sec}})$.

(ii) $E_t - E_c = \ln(\text{Tr}(X_{\text{sec}}))/\text{MU}$.

(iii) $E_t = \Phi^{-1}(\text{Tr}(X_{\text{sec}})) + E_c - E_e$, i.e., E_t is obtained from E_c by adding the Φ^{-1} -image of μ_0 , with the edge baseline cancelled.

Proof. Equivalence of (i) and (ii): Theorem ?? . Equivalence of (ii) and (iii): $\Phi^{-1}(r) = E_e + \ln(r)/\text{MU}$, so $\Phi^{-1}(\mu_0) - E_e = \ln(\mu_0)/\text{MU}$. $\square$ $\square$

Remark 3.5 (Two attempted routes and why they do not close the gap). **Route 1** (spectral weight ratio): The ρ -measure assigns total weight μ_0 to V_1 and EW weight E_c to the $\{e_1, e_2\}$ sector. The ratio is $\mu_0/E_c \approx 10.53 \neq \mu_0$. This route gives the wrong mass ratio.

Route 2 (third-generation definition): Define generation 3 as the generation whose mass ratio relative to generation 2 equals the total spectral charge μ_0 . Then (??) holds by definition. But this defers the question to the definition of generation index in the $\mathbb{Z}_3$ decomposition, rather than deriving it.

Neither route provides a derivation from the $\mathbb{Z}_3$ action on $J_3(\mathbb{O})$ to the mass formula.

4 The remaining gap for OP-B-Mass

Proposition 4.1 (Precise remaining gap). *After Addenda 44–52, the remaining single obstacle to Conjecture B is:*

OP-B-Mass (precise form). *There is no derivation showing that the $\mathbb{Z}_3$ eigenspace separation of the top quark (V_1) from the charm quark (V_{ω^2}) in $J_3(\mathbb{O})$ forces their mass ratio to equal $\text{Tr}(X_{\text{sec}}) = \mu_0$; a spectral-weight normalization argument linking the $\mathbb{Z}_3$ action on $J_3(\mathbb{O})$ to the mass formula is needed.*

Remark 4.2 (Why this is not circular). Addendum 51 established $\text{Tr}(X_{\text{sec}}) = \mu_0$ as a non-tautological theorem. OP-B-Mass asks a distinct question: why does the physical quantity m_t/m_c equal this trace? The trace identity is algebraic; the mass-ratio identification is physical. Closing OP-B-Mass requires a physical derivation—a link from the $\mathbb{Z}_3$ structure of the quark sector to the mass formula—not an algebraic restatement.

5 Status table after Addendum 52

Claim	Status after Addendum 52
Tautology in Addendum 47 resolved	Established (Addendum 51)
X_{sec} independently motivated	Established (Addendum 51, Theorem 4.1)
$\text{Tr}(X_{\text{sec}}) = \mu_0$ is a theorem	Established (Addendum 51, Corollary 4.1)
K structurally grounded	Established (Addendum 51, §5)
All D_k structurally grounded	Established
OP-B-T2: Bijection uniquely forced	Closed (Theorem ??)
Conj. A: $E_c = E_e + E_b$ structural	Established (Addendum 45, Prop. 3.1)
OP-B-Mass reduced to single claim	Established (Theorem ??)
OP-B-Mass: $m_t/m_c = \text{Tr}(X_{\text{sec}})$	Open (Prop. ??)
Conjecture B structural	Open (requires OP-B-Mass)
Conjecture D structural	Open (OP-B-Mass + $-\ln(\text{MU})/\text{MU}$ identification)

6 Summary

1. **OP-B-T2 is closed.** The $\mathbb{Z}_3$ gauge-stabilizer map (Addendum 44, Proposition 3.2) assigns $e_1 \leftrightarrow U(1)$, $e_2 \leftrightarrow SU(2)$, $e_3 \leftrightarrow G_2$ uniquely. Since each gauge group belongs to exactly one sector, the bijection $\text{edge} \leftrightarrow e_1$, $\text{boundary} \leftrightarrow e_2$, $\text{bulk} \leftrightarrow e_3$ is forced. Five of six permutations violate gauge consistency; the canonical bijection is the only survivor. This is Theorem ??.
2. **G_2 fixes all idempotents pointwise.** G_2 acts on the off-diagonal blocks and fixes V_1 entirely; no element of G_2 permutes the idempotents. The uniqueness in Theorem ?? relies on the TOE physical identification, not on the abstract G_2 action.
3. **Conjecture A is the charm anchor.** $E_c = E_e + E_b = \text{Tr}(X_{\text{sec}} \upharpoonright_{\text{EW}})$ is structural. The mass formula gives $m_c = m_e \exp(\text{MU} \pi^2) \approx 1285 \text{ MeV}$.
4. **Conjecture B reduces to a single energy-gap claim.** Given Conjecture A, Conjecture B is equivalent to $E_t - E_c = \ln(\text{Tr}(X_{\text{sec}}))/\text{MU}$ (Theorem ??). The charm saturation argument identifies the structure of this gap but does not derive it from first principles.
5. **OP-B-Mass remains open.** The single remaining obstacle is Proposition ??: there is no derivation showing that the $\mathbb{Z}_3$ eigenspace separation of top from charm forces $m_t/m_c = \text{Tr}(X_{\text{sec}})$.

References

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