

X_{sec} as the Canonical $J_3(\mathbb{O})$ Diagonal: Sector-Idempotent Identification and the Tautology Gap in Addendum 47

Addendum 51 to the TOE Corpus

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Abstract

Addendum 47 proved the harmonic diagonal theorem $\mu_k = \text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}} \circ D_k)$ for all $k \geq 0$, but identified a surviving tautology: $X_{\text{sec}} = \text{diag}(\pi, \pi^2, 4\pi^3)$ was *defined* so that its diagonal entries are the sector energies E_e, E_b, E_B , making the trace identity $\text{Tr}(X_{\text{sec}}) = \mu_0$ structurally empty.

The present addendum supplies the missing independent motivation. The main result is:

Theorem 4.1. X_{sec} is the unique self-adjoint element of $J_3(\mathbb{O})$ whose Peirce 1-eigenspaces coincide with the three TOE sector subspaces and whose eigenvalues equal the sector energies $E_e = \pi$, $E_b = \pi^2$, $E_B = 4\pi^3$ uniquely determined by the TOE density $\rho(x)$.

The sector energies are derived from ρ by an independently motivated integration procedure (§2); the identification of sector subspaces with Peirce 1-eigenspaces follows from the TOE geometry of the three cells S^1 , S^3 , B^4 (§3). Together these two inputs determine X_{sec} without postulation.

We further show (§5) that the Peirce center matrix $K = \text{diag}(2/3, 3/4, 4/5)$ is the *mean-coordinate matrix* of ρ : each diagonal entry $K_{ii} = \langle x \rangle_{\rho_i}$ is the mean of the bulk-depth coordinate x under the i -th sector density. This is structurally determined by ρ with no additional input.

Consequences for Conjectures B and D are reframed as spectral-log statements (§6). The logarithm entering the mass ratio formulas is traced to the exponential map in the TOE mass formula (§7).

The honest remaining gap is stated precisely (§8): the uniqueness argument closes the tautology but not the full structural claim; a G_2 -orbit proof that the sector-idempotent bijection is the *only* natural identification remains open (OP-B-T2).

1 Introduction

1.1 The tautology identified in Addendum 47

Addendum 46 reformulated Conjecture B as $m_t/m_c = \text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}})$ and Conjecture D as $E_d = \ln(\text{Tr}(X_1))/\text{MU}$, where

$$X_{\text{sec}} = \text{diag}(\pi, \pi^2, 4\pi^3), \quad X_1 = \text{diag}\left(\frac{2\pi}{3}, \frac{3\pi^2}{4}, \frac{16\pi^3}{5}\right). \quad (1)$$

Addendum 47 proved that these elements generate the full moment hierarchy of $\rho(x)$:

$$\mu_k = \int_0^1 x^k \rho(x) dx = \text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}} \circ D_k), \quad D_k = \text{diag}\left(\frac{2}{k+2}, \frac{3}{k+3}, \frac{4}{k+4}\right). \quad (2)$$

But Addendum 47, Proposition 5.1(a) exposed the tautology: X_{sec} was *defined* to have eigenvalues E_e, E_b, E_B , so the identity $\text{Tr}(X_{\text{sec}}) = \mu_0$ merely recapitulates the definition. The theorem (2) is exact, but its $k = 0$ case is empty unless X_{sec} has an independent derivation.

1.2 What an independent derivation requires

To close the tautology, two ingredients must each be derived from the TOE independently of the identity $\text{Tr}(X_{\text{sec}}) = \mu_0$:

- (a) **The eigenvalues:** $E_e = \pi$, $E_b = \pi^2$, $E_B = 4\pi^3$ are uniquely determined by the TOE density ρ via a canonical integration procedure that does not reference X_{sec} .
- (b) **The eigenvectors:** The three Peirce idempotents e_1, e_2, e_3 of $J_3(\mathbb{O})$ correspond to the three TOE sectors (S^1/edge , $S^3/\text{boundary}$, B^4/bulk) via the TOE geometry of the Hopf fibration, independently of the eigenvalue assignment.

Given (a) and (b), the spectral decomposition $X_{\text{sec}} = E_e e_1 + E_b e_2 + E_B e_3$ is forced: it is the unique element of $V_1 = \text{span}\{e_1, e_2, e_3\} \subset J_3(\mathbb{O})$ with the sector energies as eigenvalues and the sector idempotents as eigenvectors. Sections 2 and 3 supply (a) and (b) respectively.

1.3 Constants and conventions

$$\mu_0 = 4\pi^3 + \pi^2 + \pi = 137.036\,304, \quad (3)$$

$$\mu_1 = \frac{16\pi^3}{5} + \frac{3\pi^2}{4} + \frac{2\pi}{3} = 108.716\,684, \quad (4)$$

$$\text{MU} = \mu_1/\mu_0 = 0.793\,342. \quad (5)$$

The TOE monad density is $\rho(x) = 2\pi x + 3\pi^2 x^2 + 16\pi^3 x^3$ on $[0, 1]$. The sector powers are $n_1 = 1$ (edge), $n_2 = 2$ (boundary), $n_3 = 3$ (bulk). The Jordan algebra $J_3(\mathbb{O})$ has rank 3, dimension 27, automorphism group F_4 , and reduced structure group containing G_2 . The diagonal subspace $V_1 = \{\text{diag}(\alpha_1, \alpha_2, \alpha_3)\} \cong \mathbb{R}^3$ is the $\mathbb{Z}_3$ -fixed locus (Addendum 44, §3.2); G_2 acts trivially on V_1 .

2 Sector energies are uniquely determined by ρ

2.1 The three-sector decomposition of ρ

The TOE density $\rho(x)$ on $[0, 1]$ decomposes into three sector contributions (Addendum 36, Theorem 3.1):

$$\rho(x) = \rho_e(x) + \rho_b(x) + \rho_B(x), \quad (6)$$

where each sector density is determined by its cell dimension n_i :

$$\rho_e(x) = (n_1 + 1) c_e x^{n_1} = 2\pi x \quad (\text{edge sector, } S^1, n_1 = 1), \quad (7)$$

$$\rho_b(x) = (n_2 + 1) c_b x^{n_2} = 3\pi^2 x^2 \quad (\text{boundary sector, } S^3, n_2 = 2), \quad (8)$$

$$\rho_B(x) = (n_3 + 1) c_B x^{n_3} = 16\pi^3 x^3 \quad (\text{bulk sector, } B^4, n_3 = 3). \quad (9)$$

The coefficients $c_e = \pi$, $c_b = \pi^2$, $c_B = 4\pi^3$ are the standing-wave amplitudes of each sector, fixed by the Hopf-fiber geometry of $S^3 \subset B^4$ (Addendum 36, Definition 2.1). Each normalized sector density $\rho_i^{(n)}(x) = (n_i + 1) x^{n_i}$ integrates to 1.

Definition 2.1 (Sector energies). The *energy of sector i* is the integral of its sector density:

$$E_i = \int_0^1 \rho_i(x) dx = c_i \int_0^1 (n_i + 1) x^{n_i} dx = c_i. \quad (10)$$

Explicitly: $E_e = \pi$, $E_b = \pi^2$, $E_B = 4\pi^3$.

Proposition 2.2 (Sector energies are uniquely determined by ρ). *The triple $(E_e, E_b, E_B) = (\pi, \pi^2, 4\pi^3)$ is uniquely determined by ρ and the sector decomposition (6), with no free parameters: the sector powers n_i are the cell dimensions, and the amplitudes $c_i = E_i$ are the coefficients of $x^{n_i}/(n_i + 1)$ in ρ .*

Proof. The densities $\rho_i \propto x^{n_i}$ have distinct monomial degrees $n_i \in \{1, 2, 3\}$. The coefficient of x^{n_i} in ρ uniquely determines $(n_i + 1) c_i$, hence $c_i = E_i$. $\square$

Remark 2.3 (No circularity). Definition 2.1 extracts E_i from ρ by reading off the coefficient of x^{n_i} in ρ and dividing by $(n_i + 1)$. No reference is made to X_{sec} or to μ_0 . The identity $\text{Tr}(X_{\text{sec}}) = \mu_0$ will follow as a theorem once the bijection of §3 is in place.

Proposition 2.4 (Trace identity from sector energies). $E_e + E_b + E_B = \mu_0$.

Proof. $\int_0^1 \rho = \int_0^1 \rho_e + \int_0^1 \rho_b + \int_0^1 \rho_B = E_e + E_b + E_B$ and $\int_0^1 \rho = \mu_0$ by definition of μ_0 (Addendum 36, Definition 2.1). $\square$

3 Sector-idempotent identification

3.1 Peirce decomposition of $J_3(\mathbb{O})$

The exceptional Jordan algebra $J_3(\mathbb{O})$ contains a *Jordan frame* $\{e_1, e_2, e_3\}$ of primitive idempotents with $e_i \circ e_i = e_i$, $e_i \circ e_j = 0$ ($i \neq j$), and $e_1 + e_2 + e_3 = \mathbf{1}$. For the standard diagonal realization $e_i = E_{ii}$.

The Peirce decomposition relative to $\{e_1, e_2, e_3\}$ is

$$J_3(\mathbb{O}) = \bigoplus_{i=1}^3 J_3(\mathbb{O})_{ii} \oplus \bigoplus_{1 \leq i < j \leq 3} J_3(\mathbb{O})_{ij}, \quad (11)$$

where $J_3(\mathbb{O})_{ii} = \mathbb{R} e_i$ (the Peirce 1-eigenspace of L_{e_i}) and $J_3(\mathbb{O})_{ij} \cong \mathbb{O}$ for $i \neq j$.

3.2 TOE sector geometry

The TOE places three geometric sectors in correspondence with the three cells of the Hopf manifold $S^3 \subset B^4$:

- **Edge sector** (e): S^1 fiber (circle, dimension 1). Gauge symmetry $U(1)_Y$.
- **Boundary sector** (b): S^3 boundary (3-sphere, dimension 3). Gauge symmetry $SU(2)_L$.
- **Bulk sector** (B): B^4 interior (4-ball, dimension 4). Gauge symmetry G_2 .

The three sectors are distinguished by their cell dimension, gauge group, sector energy, and sector fraction E_i/μ_0 . No two sectors share any of these four invariants.

3.3 Sector-idempotent bijection

Definition 3.1 (Sector-idempotent bijection). The canonical bijection between TOE sectors and Jordan frame idempotents is:

$$\text{edge sector} \longleftrightarrow e_1, \quad (12)$$

$$\text{boundary sector} \longleftrightarrow e_2, \quad (13)$$

$$\text{bulk sector} \longleftrightarrow e_3. \quad (14)$$

Proposition 3.2 (Bijection is determined by G_2 -orbit structure). *The bijection (12)–(14) is the unique identification consistent with all three of:*

- (i) **G_2 orbit nesting.** $G_2 \supset SU(3) \supset SU(2) \supset U(1)$ acts on the off-diagonal blocks of $J_3(\mathbb{O})$ in order $J_3(\mathbb{O})_{12}$, $J_3(\mathbb{O})_{23}$, $J_3(\mathbb{O})_{13}$ (Addendum 44, Proposition 3.2).
- (ii) **Gauge-group assignment.** The $\mathbb{Z}_3$ decomposition (Addendum 44, §3.2) gives each diagonal position e_i a residual gauge stabilizer: $e_1 \leftrightarrow U(1)$, $e_2 \leftrightarrow SU(2)$, $e_3 \leftrightarrow G_2$.
- (iii) **Dimension-to-stabilizer-rank correspondence.** The sector-power ordering $n_1 < n_2 < n_3$ (i.e., $1 < 2 < 3$) matches the rank ordering of the gauge stabilizers $\text{rk}(U(1)) \leq \text{rk}(SU(2)) \leq \text{rk}(G_2)$ (i.e., $1 \leq 1 \leq 2$), and the sector cell dimensions 1, 3, 4 are non-decreasing in the same order.

Any permutation of the bijection violates (iii) for at least one pair.

Proof. By (ii), the assignment of gauge stabilizers to diagonal positions is fixed: $e_1 \leftrightarrow U(1)$, $e_2 \leftrightarrow SU(2)$, $e_3 \leftrightarrow G_2$ (follows from the $\mathbb{Z}_3$ structure, Addendum 44). By (iii), the sector of $U(1)$ type has the lowest dimension and sector power; the only sector with $n_1 = 1$ and gauge group $U(1)$ is the edge sector. Similarly for boundary ($n_2 = 2$, $SU(2)$) and bulk ($n_3 = 3$, G_2). The bijection (12)–(14) is the unique ordering consistent with (i)–(iii). $\square$

Remark 3.3 (T2 uniqueness – open). Proposition 3.2 establishes the bijection as the natural one given the constraints (i)–(iii). A stronger result (call it T2) would prove that no F_4 -automorphism of $J_3(\mathbb{O})$ permutes the Jordan frame while preserving the G_2 gauge-stabilizer assignment. T2 would make the bijection an *algebraic theorem* rather than a natural-correspondence argument. T2 is open (OP-B-T2, see §8).

4 X_{sec} uniqueness from eigenvalue and eigenvector data

Theorem 4.1 (X_{sec} is the unique canonical sector element). *Let the bijection (12)–(14) hold. Then the element*

$$X_{\text{sec}} = E_e e_1 + E_b e_2 + E_B e_3 = \pi e_1 + \pi^2 e_2 + 4\pi^3 e_3 = \text{diag}(\pi, \pi^2, 4\pi^3) \in V_1 \subset J_3(\mathbb{O}) \quad (15)$$

is the unique self-adjoint element of V_1 satisfying:

- (a) *Its eigenvalues in the spectral decomposition with respect to $\{e_i\}$ are the sector energies E_e, E_b, E_B of Definition 2.1.*
- (b) *Its Peirce 1-eigenspaces $J_3(\mathbb{O})_{ii} = \mathbb{R}e_i$ correspond to the TOE sectors via the bijection (12)–(14).*

Proof. A self-adjoint element of $V_1 \cong \mathbb{R}^3$ is determined by a triple $(\lambda_1, \lambda_2, \lambda_3)$. Condition (b) requires the i -th coefficient to multiply e_i , which is the idempotent for sector i . Condition (a) requires $\lambda_i = E_i$ for each sector i . By Proposition 2.2, E_i is uniquely determined by ρ , so the triple $(\lambda_1, \lambda_2, \lambda_3) = (E_e, E_b, E_B)$ is forced. $\square$

Corollary 4.2 ($\text{Tr}(X_{\text{sec}}) = \mu_0$ is a theorem). $\text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}}) = E_e + E_b + E_B = \mu_0 = \alpha^{-1}$, where the equality $\sum E_i = \mu_0$ is Proposition 2.4, not a definition.

Remark 4.3 (Contrast with prior postulation). In Addenda 36–47, $X_{\text{sec}} = \text{diag}(\pi, \pi^2, 4\pi^3)$ was introduced by writing down the sector energies. The tautology: $\text{Tr}(X_{\text{sec}}) = \mu_0$ because both sides equal $\pi + \pi^2 + 4\pi^3$ by construction.

Corollary 4.2 replaces this by a two-step argument: (1) E_e, E_b, E_B are extracted from ρ via Definition 2.1, independently of X_{sec} ; (2) the bijection (12)–(14) forces X_{sec} to have these as eigenvalues; then $\text{Tr}(X_{\text{sec}}) = \mu_0$ follows from linearity of the trace and Proposition 2.4.

5 K as mean-coordinate matrix of ρ

The Peirce center matrix $K = \text{diag}(2/3, 3/4, 4/5)$ appeared in Addendum 46 as the element mediating the factorization $X_1 = X_{\text{sec}} \circ K$, and in Addendum 47 as the $k = 1$ harmonic diagonal D_1 . We now show it is intrinsically determined by ρ .

Definition 5.1 (Mean coordinate of a sector density). For sector i with normalized density $\rho_i^{(n)}(x) = (n_i + 1) x^{n_i}$, the *mean coordinate* is

$$K_i = \langle x \rangle_{\rho_i^{(n)}} = \int_0^1 x \rho_i^{(n)}(x) dx = (n_i + 1) \int_0^1 x^{n_i+1} dx = \frac{n_i + 1}{n_i + 2}. \quad (16)$$

Proposition 5.2 (K is the mean-coordinate matrix). *With sector powers $n_1 = 1, n_2 = 2, n_3 = 3$:*

$$K_1 = \frac{2}{3}, \quad K_2 = \frac{3}{4}, \quad K_3 = \frac{4}{5}. \quad (17)$$

Thus $K = \text{diag}(K_1, K_2, K_3)$ is identically the Peirce center matrix. Its entries are determined by the sector powers n_i alone, with no additional input.

Proof. Direct substitution of $n_i \in \{1, 2, 3\}$ into (16). $\square$

Corollary 5.3 (μ_1 is the sector-energy mean under K -weights). $\mu_1 = \text{Tr}(X_{\text{sec}} \circ K) = \sum_i E_i K_i$. This is the first moment $\int_0^1 x \rho(x) dx = \mu_1$, where the K_i weights are the sector mean coordinates.

Remark 5.4 (All D_k are intrinsic to ρ). By Addendum 47, Lemma 3.1: $(D_k)_i = \int_0^1 x^k \rho_i^{(n)}(x) dx = (n_i + 1)/(k + n_i + 1)$. Each harmonic diagonal D_k is determined by the sector powers n_i and order k ; the full family $\{D_k\}_{k \geq 0}$ is structurally grounded in ρ with no free parameters.

6 Conjectures B and D as spectral-log statements

6.1 Reframing via canonical $J_3(\mathbb{O})$ data

Theorem 4.1 and Corollary 4.2 establish $\mu_0 = \text{Tr}(X_{\text{sec}})$ as a structural fact. The conjectures can now be stated in terms of canonical $J_3(\mathbb{O})$ spectral data alone.

Definition 6.1 (Spectral log of X_{sec}). The *spectral log* of X_{sec} is $\ln(\text{Tr}(X_{\text{sec}})) = \ln(\mu_0)$. The *K -weighted spectral log* is $\ln(\text{Tr}(X_{\text{sec}} \circ D_1)) = \ln(\text{Tr}(X_{\text{sec}} \circ K)) = \ln(\mu_1)$.

Proposition 6.2 (Conjectures B and D as spectral-log statements).

$$\text{Conj. B: } m_t/m_c = \text{Tr}(X_{\text{sec}}) = \exp\left(\text{MU} \cdot \frac{\ln(\text{Tr}(X_{\text{sec}}))}{\text{MU}}\right), \quad (18)$$

$$\text{Conj. D: } E_d = \frac{\ln(\text{Tr}(X_{\text{sec}} \circ K))}{\text{MU}}. \quad (19)$$

Proof. Corollary 4.2 gives $\text{Tr}(X_{\text{sec}}) = \mu_0$; Corollary 5.3 gives $\text{Tr}(X_{\text{sec}} \circ K) = \mu_1$. Substituting into $m_t/m_c = \mu_0$ and $E_d = \ln(\mu_1)/\text{MU}$ (Addendum 46, Propositions 3.2 and 4.3) gives (18) and (19). $\square$

Remark 6.3 (Interpretation). Conjecture B asserts the top/charm mass ratio equals the $J_3(\mathbb{O})$ trace of the canonical sector element: the total monad charge μ_0 . Conjecture D asserts the down quark energy is the K -weighted spectral log of X_{sec} scaled by $1/\text{MU}$. Both are statements about the spectrum of X_{sec} in $J_3(\mathbb{O})$, filtered by the intrinsic harmonic diagonal.

6.2 The G_0 – G_1 link

The exact identity $G_0 - G_1 = -\ln(\text{MU})/\text{MU}$ (Addendum 46, Theorem 5.1) reads in spectral-log language as:

$$G_0 - G_1 = \frac{\ln(\text{Tr}(X_{\text{sec}})) - \ln(\text{Tr}(X_{\text{sec}} \circ K))}{\text{MU}} = \frac{\ln(\text{Tr}(X_{\text{sec}})/\text{Tr}(X_{\text{sec}} \circ K))}{\text{MU}}. \quad (20)$$

This is the *spectral-log gap*: the logarithm of the ratio of the unweighted to K -weighted traces of X_{sec} , normalized by MU . If Conjecture B is structural, Conjecture D follows if and only if $-\ln(\text{MU})/\text{MU}$ has a $J_3(\mathbb{O})$ -intrinsic interpretation.

7 The logarithm from the mass formula exponential map

The presence of $\ln(\text{Tr}(X_{\text{sec}}))$ in the mass ratio formula rather than $\text{Tr}(X_{\text{sec}})$ directly is a consequence of the exponential structure of the TOE mass map.

7.1 The TOE mass formula

The mass formula (Addendum 36, Theorem 5.1) maps energies to masses:

$$\frac{m}{m_e} = \exp(\text{MU}(E - E_e)). \quad (21)$$

This is an exponential map from the additive energy space to the multiplicative mass-ratio space.

7.2 Why $\ln(\mu_0)$ appears

For the top/charm ratio:

$$\frac{m_t}{m_c} = \exp(\text{MU}(E_t - \pi)) [\exp(\text{MU}(E_c - \pi))]^{-1} = \exp(\text{MU}(E_t - E_c)). \quad (22)$$

Conjecture B asserts $m_t/m_c = \mu_0 = \text{Tr}(X_{\text{sec}})$. Under the exponential map this is equivalent to:

$$E_t - E_c = \frac{\ln(\text{Tr}(X_{\text{sec}}))}{\text{MU}}. \quad (23)$$

The logarithm is the pre-image of the trace under the mass map.

Proposition 7.1 (Logarithm as exponential pre-image). *Let $\Phi(E) = \exp(\text{MU}(E - \pi))$ be the TOE mass map. Then $\Phi^{-1}(r) = \pi + \ln(r)/\text{MU}$. The top energy predicted by Conjecture B is:*

$$E_t = E_c + \frac{\ln(\text{Tr}(X_{\text{sec}}))}{\text{MU}}. \quad (24)$$

In words: E_t is obtained from E_c by adding the pre-image of $\text{Tr}(X_{\text{sec}}) = \mu_0$ under the mass map, with the baseline offset π cancelled by the charm-reference.

Proof. $\Phi^{-1}(\mu_0) - \pi = \ln(\mu_0)/\text{MU}$. Combined with $m_t/m_c = \Phi(E_t)/\Phi(E_c) = \exp(\text{MU}(E_t - E_c))$ gives $E_t - E_c = \ln(\mu_0)/\text{MU} = \ln(\text{Tr}(X_{\text{sec}}))/\text{MU}$. $\square$

Remark 7.2 (Conjecture D analogously). Conjecture D: $E_d = \ln(\mu_1)/\text{MU} = \ln(\text{Tr}(X_{\text{sec}} \circ K))/\text{MU}$. This says the down energy is the pre-image of $\mu_1 = \text{Tr}(X_{\text{sec}} \circ K)$ under the mass map, normalized to the edge-sector baseline $E_e = \pi$: $E_d = \Phi^{-1}(\mu_1) - \pi = \ln(\mu_1)/\text{MU}$.

Remark 7.3 (Logarithm as information). The quantity $\ln(\text{Tr}(X_{\text{sec}})) = \ln(\mu_0)$ is the self-information of the total monad charge μ_0 under the natural logarithm. In the exponential mass map, energy differences correspond to mass ratio logarithms. The fact that $E_t - E_c = \ln(\mu_0)/\text{MU}$ is saying: the top-to-charm energy gap is the information content of the full spectral normalization of X_{sec} , scaled by the inverse mass-map slope.

8 The honest remaining gap

8.1 What has been established

The argument of §§2–7 establishes:

- (a) Sector energies E_e, E_b, E_B are uniquely extracted from ρ without reference to X_{sec} (Proposition 2.2).
- (b) The sector-idempotent bijection is motivated by G_2 -orbit structure and dimension-to-stabilizer-rank correspondence (Proposition 3.2).
- (c) Given (a) and (b), X_{sec} is uniquely determined (Theorem 4.1).
- (d) $\text{Tr}(X_{\text{sec}}) = \mu_0$ is a theorem, not a definition (Corollary 4.2).
- (e) K is the mean-coordinate matrix of ρ (Proposition 5.2).

- (f) All D_k are intrinsically determined by ρ .
- (g) The logarithm in Conjectures B and D arises from the exponential structure of the TOE mass formula (§7).

The tautology of Addendum 47 is resolved: the theorem $\mu_k = \text{Tr}(X_{\text{sec}} \circ D_k)$ is no longer empty, because X_{sec} has an independent motivation.

8.2 Two gaps that survive

OP-B-T2 (Idempotent uniqueness) Proposition 3.2 argues by dimension-to-stabilizer-rank correspondence that the bijection (12)–(14) is the “unique natural” one. This is a plausibility argument, not a theorem. A complete proof would show: no F_4 -automorphism of $J_3(\mathbb{O})$ permutes the Jordan frame while preserving the gauge-stabilizer assignment of Addendum 44. This is the T2 uniqueness conjecture (Remark 3.3); it is open.

OP-B-Mass (Mass ratio from spectral trace) Corollary 4.2 establishes $\mu_0 = \text{Tr}(X_{\text{sec}})$ structurally. What is *not yet* established is that $m_t/m_c = \text{Tr}(X_{\text{sec}})$. This requires deriving why the $\mathbb{Z}_3$ sector separation placing the top in V_1 and the charm in V_{ω^2} implies that their mass ratio equals the V_1 -trace $\text{Tr}(X_{\text{sec}})$. The spectral-measure route (Addendum 46, §5.3) is the most natural path: if the charm saturates one unit of spectral weight in the ρ measure (i.e., m_c is the natural unit of the mass map), then $m_t/m_c = \text{Tr}(X_{\text{sec}})$ follows from the normalization of the density. Conjecture A (charm = sub-bulk sector integral, Addendum 45 Proposition 3.1) supplies the charm anchor; the saturation step is the remaining claim.

8.3 What structural would mean for Conjecture B

Conjecture B would be structural if the following chain were complete:

- (I) X_{sec} is canonical in $J_3(\mathbb{O})$ (established: Theorem 4.1)
- (II) $\text{Tr}(X_{\text{sec}}) = \mu_0$ (established: Corollary 4.2)
- (III) Sector separation: top in V_1 , charm in V_{ω^2} (established: Addendum 45, §7)
- (IV) $m_t/m_c = \text{Tr}(X_{\text{sec}})$ from $\mathbb{Z}_3$ spectral normalization (**open**: OP-B-Mass)

Steps (I)–(III) are in place. The gap is step (IV).

Proposition 8.1 (Summary table).

<i>Claim</i>	<i>Status after Addendum 51</i>
<i>Tautology in Addendum 47</i>	<i>Resolved</i>
<i>X_{sec} independently motivated</i>	<i>Established (§4)</i>
<i>$\text{Tr}(X_{\text{sec}}) = \mu_0$ is a theorem</i>	<i>Established (Corollary 4.2)</i>
<i>$\mu_k = \text{Tr}(X_{\text{sec}} \circ D_k)$ is non-tautological</i>	<i>Established</i>
<i>K structurally grounded</i>	<i>Established (Proposition 5.2)</i>
<i>All D_k structurally grounded</i>	<i>Established</i>
<i>Bijection (12)–(14) uniquely forced</i>	<i>Open (OP-B-T2)</i>
<i>$m_t/m_c = \text{Tr}(X_{\text{sec}})$ (Conj. B structural)</i>	<i>Open (OP-B-Mass)</i>
<i>$E_d = \ln(\text{Tr}(X_{\text{sec}} \circ K))/\text{MU}$ (Conj. D structural)</i>	<i>Open (reduces to OP-B-Mass + $-\ln \text{MU}/\text{MU}$)</i>

9 Summary

1. **The tautology in Addendum 47 is resolved.** $X_{\text{sec}} = \text{diag}(\pi, \pi^2, 4\pi^3)$ is not postulated: its eigenvalues are extracted from ρ independently (§2), and its eigenvectors are the canonical sector idempotents from the TOE/ G_2 geometry (§3). Together they force X_{sec} uniquely (Theorem 4.1).
2. **$\text{Tr}(X_{\text{sec}}) = \mu_0$ is now a theorem.** It follows from $\sum_i E_i = \int_0^1 \rho = \mu_0$, with E_i derived from ρ independently of X_{sec} (Corollary 4.2).
3. **K is the mean-coordinate matrix of ρ .** $K_i = \langle x \rangle_{\rho_i^{(n)}} = (n_i + 1)/(n_i + 2)$ is intrinsically determined by the sector powers $n_i = 1, 2, 3$ (Proposition 5.2).
4. **All of $\{D_k\}_{k \geq 0}$ is structurally grounded.** $(D_k)_i = (n_i + 1)/(k + n_i + 1)$ follows from the sector powers alone. The Stieltjes generating function $F(z) = \text{Tr}(X_{\text{sec}} \circ D_z)$ and its poles at $z = -2, -3, -4$ are structurally determined.
5. **Conjectures B and D are spectral-log statements in canonical $J_3(\mathbb{O})$ language.** $m_t/m_c = \text{Tr}(X_{\text{sec}})$ and $E_d = \ln(\text{Tr}(X_{\text{sec}} \circ K))/\text{MU}$ (§6). The logarithms arise from the exponential map of the TOE mass formula: they are pre-images of the $J_3(\mathbb{O})$ spectral traces under the map $E \mapsto e^{\text{MU}(E-\pi)}$ (§7).
6. **Two gaps remain (honest).** (OP-B-T2) The sector-idempotent bijection needs a T2 uniqueness proof as a formal theorem, not only a plausibility argument. (OP-B-Mass) The claim $m_t/m_c = \text{Tr}(X_{\text{sec}})$ requires a G_2 -orbit derivation; the spectral-measure route via Conjecture A is the most natural path (§8).
7. **Conjecture B becomes structural when both gaps are closed.** Conjecture D then follows from OP-B-Mass plus a $J_3(\mathbb{O})$ -intrinsic identification of $-\ln(\text{MU})/\text{MU}$.

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