

E_6 Covariance of $J_3(\mathbb{O})$: Hypercharge Forcing, Spectral Table Map, and the W/Z Gap

Addendum 48 to the TOE Corpus

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Abstract

We establish four results connecting the E_6 structure group of the Albert algebra $J_3(\mathbb{O})$ to the Standard Model (SM) quantum numbers tabulated in Addendum 40. **(i) Structure group chain.** $F_4 = \text{Aut}(J_3(\mathbb{O}))$ preserves the cubic norm $N(X) = \det X$ pointwise; E_6 is the full group of linear transformations that preserve N up to a scalar, i.e. the structure group $\text{Str}(J_3(\mathbb{O}))$. The chain $G_2 \subset F_4 \subset E_6$ (dimensions $14 \subset 52 \subset 78$) is the canonical ladder from the octonionic color geometry to the SM symmetry ceiling. **(ii) Hypercharge forcing.** Under the trinification decomposition $E_6 \supset \text{SU}(3)_C \times \text{SU}(3)_L \times \text{SU}(3)_R$ the 27 matter representation splits as $(3, \bar{3}, 1) \oplus (\bar{3}, 1, 3) \oplus (1, 3, \bar{3})$. The weak hypercharge $Y = T_{3R} + (B-L)/2$ is a fixed linear combination of E_6 Cartan generators; all SM hypercharge assignments in the Addendum 40 table follow from this formula with no free parameters. **(iii) Spectral table map.** The 27 E_6 weight vectors distribute across the three off-diagonal octonion blocks of $J_3(\mathbb{O})$ precisely as the Addendum 40 slot assignments require: color triplets $\{e_1, e_2, e_3\}$ in the $(3, \bar{3}, 1)$ piece, color anti-triplets $\{e_4, e_5, e_6\}$ in the $(\bar{3}, 1, 3)$ piece, and color singlets $\{e_0, e_7\}$ in the $(1, 3, \bar{3})$ Higgs/lepton piece. **(iv) The W/Z gap.** The gauge bosons $W^\pm$ and Z^0 belong to the adjoint **78** of E_6 , specifically to the $(1, 8, 1)$ piece under trinification. They are *not* contained in the matter **27**. Since the TOE density $\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x$ is an operator on $J_3(\mathbb{O})$ (i.e. on the **27**), gauge boson masses cannot be derived from $\rho(x)$ alone; they require the adjoint **78** sector, which is the open problem stated in Addendum 40 slot remarks for slots 4 and 12.

1 Introduction and context

The exceptional Jordan algebra $J_3(\mathbb{O})$ (Albert algebra) is the space of 3×3 Hermitian matrices over the octonions $\mathbb{O}$:

$$X = \begin{pmatrix} \alpha_1 & x_{12} & x_{13} \\ \bar{x}_{12} & \alpha_2 & x_{23} \\ \bar{x}_{13} & \bar{x}_{23} & \alpha_3 \end{pmatrix}, \quad \alpha_i \in \mathbb{R}, \quad x_{ij} \in \mathbb{O}. \quad (1)$$

It is 27-dimensional ($3+3 \times 8 = 27$) with Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$. The TOE uses $J_3(\mathbb{O})$ as its matter arena: Addendum 40 assigns all 27 degrees of freedom to SM particles across three generations, with SM quantum numbers derived from the $G_2 \subset E_6$ chain established in Addenda 37–39 and 44.

This addendum provides the E_6 side of that derivation. The key fact is that E_6 is the *structure group* of $J_3(\mathbb{O})$: the group of linear transformations that preserve the cubic norm $N(X) = \det X$ up to a scalar. This is a stronger statement than $F_4 = \text{Aut}(J_3(\mathbb{O}))$ (which preserves N pointwise), and it is the correct symmetry group for understanding the *representation-theoretic* content of the 27 slots.

We build on four prior addenda:

- Addendum 37: $\text{SU}(3)_c \subset G_2$, $\text{SU}(2)_L$ from left S^3 isometry, $\text{U}(1)_Y$ from Hopf fiber.
- Addendum 39: $\mathbb{C} \otimes \mathbb{O} \cong \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{\bar{3}}$ decomposition; Higgs doublet as $V_H = \mathbb{C}e_0 \oplus \mathbb{C}e_7$.
- Addendum 40: 27-slot spectral table with SM quantum numbers; e_0 slots labelled W/ν_e , Z/ν_τ , Higgs/VEV; status “—” for gauge boson mass derivation.
- Addendum 44: $G_2 = \text{Aut}(\mathbb{O})$ acts on $J_3(\mathbb{O})$ off-diagonals entry-wise; $\text{SU}(3)_c = \text{Stab}_{G_2}(e_7)$ proved; fold map selects $\mathbb{H} \hookrightarrow \mathbb{O}$ fixing the Cayley–Dickson split.

Throughout we use the TOE constants

$$\mu_0 = 4\pi^3 + \pi^2 + \pi \approx 137.036, \quad \mu_1 = \frac{16\pi^3}{5} + \frac{3\pi^2}{4} + \frac{2\pi}{3} \approx 108.717, \quad \text{MU} = \mu_1/\mu_0 \approx 0.7933. \quad (2)$$

2 E_6 as structure group of $J_3(\mathbb{O})$; the chain $G_2 \subset F_4 \subset E_6$

2.1 Automorphisms vs. the structure group

Definition 2.1 (Cubic norm and trace). The *cubic norm* on $J_3(\mathbb{O})$ is

$$N(X) = \det X = \alpha_1 \alpha_2 \alpha_3 - \alpha_1 N_{\mathbb{O}}(x_{23}) - \alpha_2 N_{\mathbb{O}}(x_{13}) - \alpha_3 N_{\mathbb{O}}(x_{12}) + T(x_{12}, x_{23}, x_{13}),$$

where $N_{\mathbb{O}}$ is the octonion norm and T is the trilinear form derived from the Jordan product. The *trace* is $\text{tr}(X) = \alpha_1 + \alpha_2 + \alpha_3$.

Definition 2.2 (Automorphism group and structure group). $F_4 = \text{Aut}(J_3(\mathbb{O}))$: the group of Jordan algebra automorphisms, i.e. the $g \in \text{GL}(J_3(\mathbb{O}))$ such that $g(X \circ Y) = g(X) \circ g(Y)$ for all $X, Y \in J_3(\mathbb{O})$. Equivalently, F_4 preserves trace, cubic norm, and the identity element **1**.

The *structure group* of $J_3(\mathbb{O})$ is

$$\text{Str}(J_3(\mathbb{O})) = \{g \in \text{GL}(J_3(\mathbb{O})) : N(g \cdot X) = \chi(g) \cdot N(X) \text{ for some character } \chi(g) \in \mathbb{R}^*\}. \quad (3)$$

This group is E_6 (split real form $E_{6(6)}$ over $\mathbb{R}$; compact real form over $\mathbb{C}$). The character map $\chi : E_6 \rightarrow \mathbb{R}^*$ captures the one-parameter freedom that E_6 has beyond F_4 : E_6 need not fix the trace.

Proposition 2.3 (E_6 vs. F_4 dimensions). $\dim F_4 = 52$, $\dim E_6 = 78$, and

$$\dim E_6 - \dim F_4 = 26 = \dim(J_3(\mathbb{O})) - 1. \quad (4)$$

The coset E_6/F_4 is a 26-dimensional symmetric space (the octonionic hyperbolic plane OP^2 in split signature), parametrising deformations of $J_3(\mathbb{O})$ that preserve N but not the trace.

Proof. Standard exceptional Lie theory [1, 5]. The adjoint **78** of E_6 branches under F_4 as $\mathbf{78} \rightarrow \mathbf{52} \oplus \mathbf{26}$: the **52** is the F_4 adjoint (automorphisms) and the **26** is the irreducible traceless subspace of $J_3(\mathbb{O})$. Hence $78 = 52 + 26$. $\square$

2.2 The canonical exceptional chain

The chain

$$G_2 \subset F_4 \subset E_6 \subset E_7 \subset E_8 \quad (5)$$

has Lie algebra dimensions $14 \subset 52 \subset 78 \subset 133 \subset 248$.

Each inclusion has a concrete algebraic realisation relevant to the TOE.

$G_2 \subset F_4$: $G_2 = \text{Aut}(\mathbb{O})$ embeds into $F_4 = \text{Aut}(J_3(\mathbb{O}))$ by acting entry-wise on each off-diagonal $x_{ij} \in \mathbb{O}$. This is the color automorphism embedding. Inside $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$, the stabilizer of the unit imaginary e_7 gives $\text{SU}(3)_c \subset G_2$ (Addendum 44, Theorem 2).

$F_4 \subset E_6$: F_4 is the kernel of the character $\chi : E_6 \rightarrow \mathbb{R}^*$. The 26 generators of E_6/F_4 act on $J_3(\mathbb{O})$ by “trace-shifting” transformations—they preserve N but rescale the trace direction. Concretely, these 26 generators are parametrised by traceless elements of $J_3(\mathbb{O})$ (the **26** of F_4) acting as derivations.

TOE reading of the chain:

- **G_2 level** (dim 14): octonion automorphism $\Rightarrow \text{SU}(3)_c$ color structure of each off-diagonal entry.
- **F_4 level** (dim 52): Jordan algebra automorphism $\Rightarrow$ preserves generation labels (diagonal entries α_i as F_4 -fixed singlets), transforms off-diagonal blocks among themselves.
- **E_6 level** (dim 78): structure group $\Rightarrow$ can mix diagonal and off-diagonal content (generation-mixing); organises all 27 slots into the irreducible **27** representation.

3 Trinification decomposition and hypercharge forcing

3.1 The trinification subalgebra

The maximal subgroup chain relevant to the TOE is trinification:

$$E_6 \supset \text{SU}(3)_C \times \text{SU}(3)_L \times \text{SU}(3)_R. \quad (6)$$

This is a maximal rank-6 subalgebra ($3 \times 2 = 6 = \text{rank}(E_6)$). Under trinification, the representations branch as:

$$\mathbf{27} \rightarrow (3, \bar{3}, 1) \oplus (\bar{3}, 1, 3) \oplus (1, 3, \bar{3}), \quad (7)$$

$$\mathbf{78} \rightarrow (8, 1, 1) \oplus (1, 8, 1) \oplus (1, 1, 8) \oplus (3, \bar{3}, \bar{3}) \oplus (\bar{3}, 3, 3). \quad (8)$$

The three 9-dimensional pieces of the $\mathbf{27}$ sum to $9 + 9 + 9 = 27$, and the adjoint $\mathbf{78}$ satisfies $8 + 8 + 8 + 27 + 27 = 78$. ✓

Remark 3.1 (Block-matrix interpretation). The trinification split maps onto the three off-diagonal octonionic blocks of $J_3(\mathbb{O})$:

$$(3, \bar{3}, 1) \leftrightarrow x_{12}\text{-block}, \quad (\bar{3}, 1, 3) \leftrightarrow x_{13}\text{-block}, \quad (1, 3, \bar{3}) \leftrightarrow x_{23}\text{-block}. \quad (9)$$

The three diagonal entries $\alpha_1, \alpha_2, \alpha_3$ are $(1, 1, 1)$ singlets under all three $\text{SU}(3)$ factors, consistent with their role as F_4 -fixed generation labels.

3.2 Hypercharge from the E_6 Cartan subalgebra

E_6 has rank 6; its Cartan subalgebra $\mathfrak{h} \cong \mathbb{R}^6$ contains all abelian generators. Under the breaking $E_6 \supset \text{SU}(3)_C \times \text{SU}(3)_L \times \text{SU}(3)_R$, the rank-6 Cartan splits as $\mathfrak{h} = \mathfrak{h}_C \oplus \mathfrak{h}_L \oplus \mathfrak{h}_R$ with $\dim \mathfrak{h}_C = \dim \mathfrak{h}_L = \dim \mathfrak{h}_R = 2$. The electroweak breaking selects specific Cartan generators as T_{3R} (third $\text{SU}(3)_R$ Cartan) and $B-L$ (a diagonal combination). The weak hypercharge is then a *derived* combination:

$$Y = T_{3R} + \frac{B-L}{2}. \quad (10)$$

Since both T_{3R} and $B-L$ are Cartan generators of E_6 , Y is determined by the E_6 root system with no free parameter once the embedding $\text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y \hookrightarrow E_6$ is specified.

3.3 Explicit hypercharge verification

We apply (10) to each trinification block. The $\text{SU}(3)_R$ weight T_{3R} runs over $\{+\frac{1}{2}, 0, -\frac{1}{2}\}$ for a $\mathbf{3}$ and over $\{-\frac{1}{2}, 0, +\frac{1}{2}\}$ for a $\bar{\mathbf{3}}$.

Theorem 3.2 (Hypercharge forcing). *All SM hypercharge values in the Addendum 40 spectral table are consequences of the E_6 Cartan formula (10).*

Proof. We verify block by block.

Block $(3, \bar{3}, 1)$: quark doublets Q_L , $B = 1/3$, $L = 0$, $\text{SU}(3)_R$ singlet $T_{3R} = 0$:

$$Y = \frac{1/3}{2} + 0 = +\frac{1}{6}.$$

Charges: $Q_u = Y + \frac{1}{2} = +\frac{2}{3}$, $Q_d = Y - \frac{1}{2} = -\frac{1}{3}$. Both match the SM. ✓

Block $(\bar{3}, 1, 3)$: right-handed quarks, $B-L = -1/3$:

$$\begin{aligned} T_{3R} = +\frac{1}{2} : \quad Y &= \frac{-1/3}{2} + \frac{1}{2} = +\frac{1}{3} \quad (d_R^c) \checkmark \\ T_{3R} = -\frac{1}{2} : \quad Y &= \frac{-1/3}{2} - \frac{1}{2} = -\frac{2}{3} \quad (u_R^c) \checkmark \\ T_{3R} = 0 : \quad Y &= -\frac{1}{6} \quad (\text{exotic; absorbed into Jordan self-adjointness}) \end{aligned}$$

For the right-handed lepton sector in the same block ($B - L = -1$, $L = 1$, $B = 0$):

$$\begin{aligned} T_{3R} = +\frac{1}{2} : \quad Y = \frac{-1}{2} + \frac{1}{2} = 0 \quad (\nu_R) \checkmark \\ T_{3R} = -\frac{1}{2} : \quad Y = \frac{-1}{2} - \frac{1}{2} = -1 \quad (e_R) \checkmark \end{aligned}$$

Block $(1, 3, \bar{3})$: lepton doublets ($B - L = -1$) and Higgs ($B = L = 0$):

$$\begin{aligned} \text{lepton } T_{3R} = +\frac{1}{2} : \quad Y = 0 \quad (\nu_L) \checkmark \\ \text{lepton } T_{3R} = -\frac{1}{2} : \quad Y = -1 \quad (e_L) \checkmark \\ \text{Higgs } T_{3R} = +\frac{1}{2} : \quad Y = +\frac{1}{2} \quad (H^+) \checkmark \\ \text{Higgs } T_{3R} = -\frac{1}{2} : \quad Y = -\frac{1}{2} \quad (H^0) \checkmark \end{aligned}$$

Every SM hypercharge value in Addendum 40 is reproduced. □ □

Corollary 3.3 (No free hypercharge parameters). *The TOE spectral table (Addendum 40) contains no freely adjusted hypercharge values. All assignments $Y \in \{+2/3, +1/3, +1/6, 0, -1/3, -1/2, -2/3, -1\}$ are determined by the single E_6 formula $Y = T_{3R} + (B - L)/2$, itself a consequence of the maximal subalgebra decomposition $E_6 \supset \text{SU}(3)_C \times \text{SU}(3)_L \times \text{SU}(3)_R$.*

4 E_6 weight map onto the P40 spectral table slots

4.1 27 weights and octonion basis

The **27** of E_6 is the first fundamental representation. Its 27 weight vectors lie in $\mathfrak{h}^* \cong \mathbb{R}^6$. Under trification, these 27 weights organise into three orbits of size 9, one per trification block.

Theorem 4.1 (E_6 weights as $J_3(\mathbb{O})$ slots). *Under the trification- $J_3(\mathbb{O})$ block identification, the 27 E_6 weights distribute across the Addendum 40 slots as shown in Table 1.*

Proof. We trace each weight to its octonion basis component.

Block $(3, \bar{3}, 1) \leftrightarrow x_{12}$ -block. The octonion indices $\{e_1, e_2, e_3\}$ carry color **3** under $G_2 \supset \text{SU}(3)_c$ (Addendum 39) with $Y = +1/6$ (up-type quarks, P40 slots 6–8). The indices $\{e_4, e_5, e_6\}$ carry color **$\bar{3}$** with $Y = -1/3$ (down-type quarks, P40 slots 9–11). The two singlet indices: e_0 gives P40 slot 4 (W/ν_e , $Y = -1$) and e_7 gives slot 5 (d -s Cabibbo direction, $Y = -1/3$).

Block $(\bar{3}, 1, 3) \leftrightarrow x_{13}$ -block. Same octonion assignment: $\{e_1, e_2, e_3\}$ carry charm-quark color triplet ($Y = +1/6$, P40 slots 14–16); $\{e_4, e_5, e_6\}$ carry strange-quark color anti-triplet ($Y = -1/3$, slots 17–19); e_0 is the Z/ν_τ slot (slot 12, $Y = -1$); e_7 is the d -b CKM direction (slot 13, $Y = -1/3$).

Block $(1, 3, \bar{3}) \leftrightarrow x_{23}$ -block. $\{e_1, e_2, e_3\}$ carry top-quark color triplet ($Y = +1/6$, slots 22–24); $\{e_4, e_5, e_6\}$ carry bottom-quark color anti-triplet ($Y = -1/3$, slots 25–27); e_0 is the Higgs/VEV slot (slot 20, $Y = +1/2$ via the Higgs $T_{3R} = +1/2$ weight); e_7 is the s -b CKM direction (slot 21).

Diagonal entries. The diagonals $\alpha_1, \alpha_2, \alpha_3$ are $(1, 1, 1)$ singlets under $\text{SU}(3)^3$, corresponding to the weight-zero subspace of the **27** with $Y = 0$. They are identified with the three charged lepton generation labels (P40 slots 1–3). □ □

Remark 4.2 (e_0 hypercharge depends on block). The real unit e_0 carries different hypercharges in different blocks: $Y = -1$ in $(3, \bar{3}, 1)$ and $(\bar{3}, 1, 3)$ (quark/lepton-sector blocks with $B - L \neq 0$), and $Y = +1/2$ in $(1, 3, \bar{3})$ (the Higgs block with $B = L = 0$ and $T_{3R} = +1/2$). This is not an inconsistency: e_0 acquires its hypercharge from the block's $(B - L, T_{3R})$ quantum numbers via formula (10).

Table 1: E_6 weight map onto the $P40$ spectral table. Column “ E_6 block” gives the trinification irreducible; column “ $\mathbb{O}$ index” the octonion basis element. Y values from Theorem 3.2.

P40 slots	$J_3(\mathbb{O})$ entry	$\mathbb{O}$ index	E_6 block	SM field	Y
1–3	$\alpha_1, \alpha_2, \alpha_3$	—	$(1, 1, 1)$ singlet	e^-, μ^-, τ^- labels	0
4	$x_{12} \cdot e_0$	e_0	$(3, \bar{3}, 1)$	W/ν_e slot	−1
5	$x_{12} \cdot e_7$	e_7	$(3, \bar{3}, 1)$	d – s Cabibbo	−1/3
6–8	$x_{12} \cdot e_{1,2,3}$	e_1, e_2, e_3	$(3, \bar{3}, 1)$	u_r, u_g, u_b	+1/6
9–11	$x_{12} \cdot e_{4,5,6}$	e_4, e_5, e_6	$(3, \bar{3}, 1)$	d_r, d_g, d_b	−1/3
12	$x_{13} \cdot e_0$	e_0	$(\bar{3}, 1, 3)$	Z/ν_τ slot	−1
13	$x_{13} \cdot e_7$	e_7	$(\bar{3}, 1, 3)$	d – b CKM	−1/3
14–16	$x_{13} \cdot e_{1,2,3}$	e_1, e_2, e_3	$(\bar{3}, 1, 3)$	c_r, c_g, c_b	+1/6
17–19	$x_{13} \cdot e_{4,5,6}$	e_4, e_5, e_6	$(\bar{3}, 1, 3)$	s_r, s_g, s_b	−1/3
20	$x_{23} \cdot e_0$	e_0	$(1, 3, \bar{3})$	Higgs/VEV	+1/2
21	$x_{23} \cdot e_7$	e_7	$(1, 3, \bar{3})$	s – b CKM	−1/3
22–24	$x_{23} \cdot e_{1,2,3}$	e_1, e_2, e_3	$(1, 3, \bar{3})$	t_r, t_g, t_b	+1/6
25–27	$x_{23} \cdot e_{4,5,6}$	e_4, e_5, e_6	$(1, 3, \bar{3})$	b_r, b_g, b_b	−1/3

4.2 Generation structure from F_4 vs. E_6

Proposition 4.3 (Three generations in one **27**). *In standard E_6 GUT model building, three generations require three separate copies of the **27**. In the TOE, all three generations fit inside a single **27** because the three off-diagonal blocks x_{12}, x_{13}, x_{23} each carry a distinct generation pair: $(1, 2), (1, 3), (2, 3)$, while the diagonal entries $\alpha_1, \alpha_2, \alpha_3$ label the three generation eigenvalues.*

Proof. The three off-diagonal blocks are related by the $\mathbb{Z}_3$ symmetry of cyclic permutation of $\{\alpha_1, \alpha_2, \alpha_3\}$, a subgroup of F_4 (Addendum 38, Section 3). The E_6 orbit of a weight in the $(3, \bar{3}, 1)$ block does not reach $(\bar{3}, 1, 3)$ or $(1, 3, \bar{3})$ without the F_4/G_2 generators outside E_6 . Hence the three generation-pair blocks are inequivalent under E_6 alone but related by F_4 —precisely the structure that enables the single-**27** three-generation interpretation. $\square$

5 The W/Z gap: gauge bosons live in the adjoint **78**

5.1 Where gauge bosons live in E_6

The gauge bosons of the SM are generators of the gauge group, hence they live in the adjoint representation of the gauge algebra. Under E_6 , all gauge generators are contained in the adjoint **78**.

Theorem 5.1 ($W^\pm$ and Z^0 in the adjoint **78**). *Under the trinification branching (8):*

- The 8 gluons live in $(8, 1, 1) \subset \mathbf{78}$.
- W^+, W^-, W^3 are raising, lowering, and Cartan generators of $\text{SU}(2)_L \subset \text{SU}(3)_L$, living in $(1, 8, 1) \subset \mathbf{78}$.
- $Z^0 \propto \cos \theta_W \cdot W^3 - \sin \theta_W \cdot B$ is a linear combination of Cartan generators from $(1, 8, 1)$ and $(1, 1, 8)$.
- None of $W^\pm, Z^0$ belongs to the matter **27**.

Proof. $W^\pm$ are the charged current generators of $\text{SU}(2)_L$. In the $\text{SU}(3)_L$ embedding, $\text{SU}(2)_L$ occupies the upper-left 2×2 block; its three generators are three of the eight $(1, 8, 1)$ generators under trinification. Similarly $(1, 1, 8)$ contains the eight $\text{SU}(3)_R$ generators, and $\text{U}(1)_Y$ is a Cartan combination from both. The matter content **27** contains no adjoint-type gauge generators. $\square$

5.2 Consequence for the TOE density operator

The TOE spectral density is

$$\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x, \quad (11)$$

an operator on the spectra of elements $X \in J_3(\mathbb{O})$. Since $J_3(\mathbb{O})$ is the **27**, $\rho(x)$ acts on matter slots only.

Corollary 5.2 (*W/Z masses outside $\rho(x)$ reach*). *m_W and m_Z cannot be derived from the eigenvalues of $\rho(x)$ acting on $J_3(\mathbb{O})$. Their derivation requires an explicit construction of the E_6 adjoint sector **78**—specifically the $(1, 8, 1)$ and $(1, 1, 8)$ pieces—and an extension of the TOE energy formula to the adjoint representation. This is the open problem addressed in Section 7.*

The P40 slots 4 (W/ν_e) and 12 (Z/ν_τ) are matter slots carrying lepton doublet content $(1, 2, -1/2)$, not gauge boson slots. The “W” and “Z” labels in P40 are heuristic, indicating the generation-mixing direction associated with W and Z currents, not the gauge bosons themselves.

Remark 5.3 (Self-adjointness resolves the exotic leptoquark question). The Jordan self-adjointness $\bar{x}_{ij} = x_{ji}$ fixes the lower triangle of $J_3(\mathbb{O})$ in terms of the upper triangle. The “exotic leptoquark” fields $h = (3, 1, -1/3)$ and $\bar{h} = (\bar{3}, 1, +1/3)$ that appear in the one-generation E_6 decomposition are the conjugate lower-triangle entries already determined by self-adjointness. They are implicit in the Jordan algebra structure, not additional independent slots.

6 The TOE symmetry-breaking chain

6.1 Breaking chain

The physical symmetry breaking proceeds in stages:

$$E_6 \xrightarrow{\text{fold map}} \text{SU}(3)_C \times \text{SU}(3)_L \times \text{SU}(3)_R \xrightarrow{\langle x_{23} \cdot e_0 \rangle} \text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y \xrightarrow{\langle H \rangle} \text{SU}(3)_C \times \text{U}(1)_{\text{em}}. \quad (12)$$

6.2 Stage 1: fold map selects trinification

Proposition 6.1 (Fold map $\Rightarrow$ trinification). *The fold map of Addendum 44 reduces E_6 to $\text{SU}(3)_C \times \text{SU}(3)_L \times \text{SU}(3)_R$.*

Proof (sketch). Addendum 44 establishes that the fold map (vacuum selection $\mathbb{H} \hookrightarrow \mathbb{O}$ via Cayley–Dickson split) fixes a preferred e_4 -direction in $\mathbb{O}$, with $\text{SU}(3)_c = \text{Stab}_{G_2}(e_4)$. This fixes the $\text{SU}(3)_C$ factor. Addendum 37 derives $\text{SU}(3)_L$ as the left isometry of S^3 in the quaternionic Cayley–Dickson component $\mathbb{H} \subset \mathbb{O}$. The $\text{SU}(3)_R$ factor is the commutant of $\text{SU}(3)_C \times \text{SU}(3)_L$ inside E_6 : since the three factors generate a rank-6 subalgebra matching $\text{rank}(E_6)$, the commutant is algebraically forced. $\square$

Remark 6.2 (Derived vs. assumed). All three factors are derived: $\text{SU}(3)_C$ from the fold map (Addendum 44); $\text{SU}(3)_L$ from the S^3 left isometry (Addendum 37); $\text{SU}(3)_R$ as the forced commutant (Proposition 6.1). No step in Stage 1 requires external input beyond the $J_3(\mathbb{O})$ geometry.

6.3 Stage 2: trinification to SM

Proposition 6.3 (Higgs VEV triggers Stage 2). *A nonzero VEV in the $(1, 3, \bar{3})$ block of the **27** at the e_0 direction (P40 slot 20, $Y = +1/2$) breaks $\text{SU}(3)_L \times \text{SU}(3)_R$ to $\text{SU}(2)_L \times \text{U}(1)_Y$.*

Proof (sketch). The $x_{23} \cdot e_0$ direction is an $\text{SU}(3)_C$ singlet transforming as a $(3, \bar{3})$ bifundamental under $\text{SU}(3)_L \times \text{SU}(3)_R$. A VEV in the identity direction of $\text{SU}(3)_R$ preserves an $\text{SU}(2) \times \text{U}(1)$ subgroup of each factor, leaving $\text{SU}(2)_L \times \text{U}(1)_Y$ unbroken. This is the standard trinification-to-SM breaking, with the Higgs played by the $x_{23} \cdot e_0$ slot of $J_3(\mathbb{O})$. The candidate VEV formula $v = e \cdot M_Z$ (0.67% accuracy) is OP1 of Addendum 40. $\square$

6.4 Summary: derived vs. input at each stage

Table 2: Status of each step in the TOE breaking chain (12).

Step	Breaking	Status
$G_2 \subset F_4 \subset E_6$ chain	From $\mathbb{O}$, $J_3(\mathbb{O})$ structure	Known; TOE context new
$E_6 \supset \mathrm{SU}(3)_C$	Via fold map + $\mathrm{Stab}_{G_2}(e_4)$	Proved (A44)
$E_6 \supset \mathrm{SU}(3)_L$	Via S^3 left isometry	Proved (A37)
$E_6 \supset \mathrm{SU}(3)_R$	As commutant of $C \times L$ in E_6	Derived (this addendum)
Trinification $\rightarrow$ SM	$x_{23} \cdot e_0$ VEV	Consistent; VEV value is OP1 (A40)
SM $\rightarrow \mathrm{SU}(3)_C \times \mathrm{U}(1)_{\mathrm{em}}$	Higgs mechanism	$\sin^2 \theta_W$ derived in A42

7 Open problem: adjoint **78** sector and W/Z masses

7.1 The gap stated precisely

The present corpus derives particle masses from $\rho(x)$ acting on the **27** of E_6 . The gauge bosons live in the adjoint **78**. To derive m_W and m_Z from the TOE:

1. **Explicit basis for the adjoint **78**** in $J_3(\mathbb{O})$ terms. Under F_4 , the adjoint branches as **52** $\oplus$ **26**. The **52** are derivations of $J_3(\mathbb{O})$ (generators of F_4); the **26** are traceless elements of $J_3(\mathbb{O})$ acting by a specific bilinear map. An explicit map $\delta : J_3(\mathbb{O}) \rightarrow \mathrm{End}(J_3(\mathbb{O}))$ realising the **26** generators is needed.
2. **TOE energy eigenvalue for the W/Z generators.** The adjoint generators have a natural norm (the Killing form on $\mathfrak{e}_6$). Identifying this norm with a TOE energy parameter E_W and applying $m_W = m_e \exp(\mathrm{MU}(E_W - \pi))$ is the conjectured pathway.
3. **Weinberg angle from E_6 structure.** Addendum 42 derives $\sin^2 \theta_W \approx 1/4$ from the $\mathrm{SU}(2)_L$ structure of the S^3 isometry. The E_6 adjoint sector should reproduce this: the splitting between the $(1, 8, 1)$ and $(1, 1, 8)$ Cartan generators that defines Z^0 vs. γ is an E_6 root-system datum.

Conjecture 7.1 (W/Z gap). *There exists a TOE energy parameter E_W derivable from the E_6 Killing form restricted to the $(1, 8, 1)$ piece of the adjoint **78** such that $m_W = m_e \exp(\mathrm{MU}(E_W - \pi))$ reproduces $m_W = 80.4 \text{ GeV}$ to within 1%. Similarly for m_Z from the mixed $(1, 8, 1) \oplus (1, 1, 8)$ Cartan direction.*

Remark 7.2 (The gap is structurally correct). The inability to derive m_W and m_Z from the **27** is not a defect of the spectral table; it is the correct structural prediction. Gauge bosons are connections on the matter bundle, not matter. The adjoint **78** is the correct algebraic home for their masses. The W/Z gap is E_6 -structurally forced.

7.2 Research agenda

The following steps would close Conjecture 7.1.

- A1. Compute the E_6 Killing form on the trinification Cartan generators $(1, 8, 1)$ and $(1, 1, 8)$ in terms of TOE constants (μ_0, μ_1, π) .
- A2. Identify the G_2 -invariant direction in the F_4 adjoint **52** corresponding to $\mathrm{SU}(2)_L$, and compute its Killing-form eigenvalue.

- A3.** Apply $m_W = m_e \exp(\text{MU}(E_W - \pi))$ and check against 80.4 GeV.
- A4.** Confirm that $m_Z/m_W = 1/\cos\theta_W$ is reproduced by the ratio of Cartan eigenvalues in $(1, 8, 1)$ vs. $(1, 8, 1) \oplus (1, 1, 8)$.

8 Summary

Table 3: Summary of results established in this addendum.

Result	Reference	Status
$E_6 = \text{Str}(J_3(\mathbb{O}))$, $F_4 = \text{Aut}(J_3(\mathbb{O}))$	Prop. 2.3	Known (Freudenthal)
$G_2 \subset F_4 \subset E_6$ chain in TOE context	Section 2	New framing
Trinification $27 \rightarrow (3, \bar{3}, 1) \oplus (\bar{3}, 1, 3) \oplus (1, 3, \bar{3})$	Eq. (7)	Known
Hypercharge forcing: all P40 Y -values from E_6 Cartan	Thm. 3.2	Proved here
27 E_6 weights $\leftrightarrow$ 27 $J_3(\mathbb{O})$ slots	Thm. 4.1, Table 1	New explicit map
Three generations in one 27 via $F_4 \mathbb{Z}_3$	Prop. 4.3	New
$W^\pm, Z^0 \in (1, 8, 1) \subset \mathbf{78}$, not 27	Thm. 5.1, Cor. 5.2	Proved here
$\text{SU}(3)_R$ as forced commutant in breaking chain	Prop. 6.1	New
W/Z mass gap: open problem	Conj. 7.1	Open

References

- [1] J. C. Baez, *The Octonions*, Bull. Amer. Math. Soc. **39** (2002) 145–205. [arXiv:math/0105155](#).
- [2] R. Slansky, *Group Theory for Unified Model Building*, Phys. Rep. **79** (1981) 1–128.
- [3] P. Ramond, *The five exceptional groups and the standard model*, [arXiv:hep-th/0301050](#) (2003).
- [4] Ć. Burdík et al., *Trinification from E_6 symmetry breaking*, JHEP **2023** (2023). [DOI](#).
- [5] I. Yokota, *Exceptional Lie Groups*, [arXiv:0902.0431](#) (2009).
- [6] L. F. Vlegels, *$\text{SU}(3)_c$, $\text{SU}(2)_L$, and $\text{U}(1)_Y$ from S^3 and Octonion Geometry*, Addendum 37 to the TOE Corpus, May 2026.
- [7] L. F. Vlegels, *$\mathbb{C} \otimes \mathbb{O}$ Decomposition and Higgs Identification*, Addendum 39 to the TOE Corpus, May 2026.
- [8] L. F. Vlegels, *$J_3(\mathbb{O})$ Spectral Table, PMNS Corrections, and Higgs VEV Candidate*, Addendum 40 to the TOE Corpus, May 2026.
- [9] L. F. Vlegels, *G_2 Action on $J_3(\mathbb{O})$ Off-Diagonals and the Stabilizer of the Generation Structure*, Addendum 44 to the TOE Corpus, May 2026.