

Route 3 Resolved: $\rho(x)$ as Sector Trace Generating Function Spectral Analysis of $L_{X_{\text{sec}}}$ and the Harmonic Diagonal

Addendum 47 to the TOE Corpus

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Abstract

Addendum 46 identified Route 3—treating $\rho(x)$ as the spectral density of X_{sec} under a $J_3(\mathbb{O})$ probability measure—as the most natural path toward closing Conjectures B and D. The present addendum executes Route 3 in full.

Negative result. $\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x$ is *not* the spectral density of $L_{X_{\text{sec}}}$ in any standard operator sense. The Peirce spectrum of $L_{X_{\text{sec}}}$ is a discrete set in $[\pi, 4\pi^3]$, entirely disjoint from $[0, 1]$; the spectral trace equals $9\mu_0$, not μ_0 ; and the Gauss quadrature nodes of ρ do not match the Peirce eigenvalues under any natural normalization.

Positive result (new theorem). $\rho(x)$ is the *sector trace generating polynomial* of X_{sec} . For all $k \geq 0$:

$$\mu_k = \int_0^1 x^k \rho(x) dx = \text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}} \circ D_k), \quad D_k = \text{diag}\left(\frac{2}{k+2}, \frac{3}{k+3}, \frac{4}{k+4}\right).$$

The matrix D_k is the *harmonic diagonal of order k* . At $k = 0$, $D_0 = I$ and $\mu_0 = \text{Tr}(X_{\text{sec}})$. At $k = 1$, $D_1 = K$ is the Peirce center matrix of Addendum 46. The generating function $F(z) = \text{Tr}(X_{\text{sec}} \circ D_z)$ is rational with poles at $z = -2, -3, -4$. A further exact identity: $\text{Tr}_{\text{spec}}(L_{X_{\text{sec}}})/\dim J_3(\mathbb{O}) = \mu_0/3$.

Status. Conjectures B and D are not closed. The trace identity $\mu_0 = \text{Tr}(X_{\text{sec}})$ is tautological. The structural gap (OP-B) survives.

1 Introduction

1.1 Context

Addendum 46 reformulated Conjectures B and D as $J_3(\mathbb{O})$ trace identities and listed three routes toward structural derivations. Route 3 was declared “most natural” with the informal description: treat $\rho(x)$ as the spectral density of X_{sec} under a $J_3(\mathbb{O})$ probability measure, so that its normalization $\int_0^1 \rho(x) dx = \mu_0 = \text{Tr}(X_{\text{sec}})$ is the trace formula for the spectral measure.

This addendum makes Route 3 precise and resolves both questions:

1. Do the Gauss quadrature nodes of ρ match Peirce eigenvalues of $L_{X_{\text{sec}}}$?
2. What is the correct $J_3(\mathbb{O})$ algebraic object encoding ρ and its full moment hierarchy?

1.2 Constants and notation

$$\mu_0 = 4\pi^3 + \pi^2 + \pi = 137.036\,304, \tag{1}$$

$$\mu_1 = \frac{16\pi^3}{5} + \frac{3\pi^2}{4} + \frac{2\pi}{3} = 108.716\,684, \tag{2}$$

$$\text{MU} = \mu_1/\mu_0 = 0.793\,342. \tag{3}$$

The sector element $X_{\text{sec}} = \text{diag}(\pi, \pi^2, 4\pi^3) \in V_1 \subset J_3(\mathbb{O})$ has eigenvalues $\alpha_1 = \pi$, $\alpha_2 = \pi^2$, $\alpha_3 = 4\pi^3$ with sector powers $n_1 = 1$, $n_2 = 2$, $n_3 = 3$. The monad density is $\rho(x) = 2\pi x + 3\pi^2 x^2 + 16\pi^3 x^3$ with moments $\mu_k = 16\pi^3/(k+4) + 3\pi^2/(k+3) + 2\pi/(k+2)$.

2 Spectral analysis of $L_{X_{\text{sec}}}$

2.1 Peirce spectrum

The Jordan multiplication operator $L_{X_{\text{sec}}} : J_3(\mathbb{O}) \rightarrow J_3(\mathbb{O})$ acts by $L_{X_{\text{sec}}}(Y) = X_{\text{sec}} \circ Y$. For a diagonal element $X_{\text{sec}} = \sum_i \alpha_i e_i$ with distinct eigenvalues, the Peirce decomposition gives (McCrimmon [4]):

- Eigenvalue α_i on $\mathbb{R}e_i \subset V_1$, multiplicity 1;

Table 1: Peirce spectrum of $L_{X_{\text{sec}}}$ on $J_3(\mathbb{O})$.

Eigenvalue	Multiplicity	Numerical value
$\alpha_1 = \pi$	1	3.14159
$\alpha_2 = \pi^2$	1	9.86960
$\alpha_3 = 4\pi^3$	1	124.025
$(\alpha_1 + \alpha_2)/2$	8	6.5056
$(\alpha_1 + \alpha_3)/2$	8	63.583
$(\alpha_2 + \alpha_3)/2$	8	66.947

- Eigenvalue $(\alpha_i + \alpha_j)/2$ on $J_{ij} \cong \mathbb{O}$, multiplicity 8, for each $i < j$.

The six distinct Peirce eigenvalues are given in Table 1.

Total weight: $1 + 1 + 1 + 8 + 8 + 8 = 27 = \dim J_3(\mathbb{O})$. The spectrum lies in $[\pi, 4\pi^3] \approx [3.14, 124.0]$, entirely disjoint from $[0, 1]$.

Proposition 2.1 (Spectral trace identity).

$$\text{Tr}_{\text{spec}}(L_{X_{\text{sec}}}) = \sum_i w_i \lambda_i = 9\mu_0, \quad \frac{\text{Tr}_{\text{spec}}(L_{X_{\text{sec}}})}{\dim J_3(\mathbb{O})} = \frac{\mu_0}{3}. \quad (4)$$

Proof. Direct expansion using Table 1:

$$\begin{aligned} \text{Tr}_{\text{spec}} &= (\alpha_1 + \alpha_2 + \alpha_3) + 8 \cdot \frac{\alpha_1 + \alpha_2}{2} + 8 \cdot \frac{\alpha_1 + \alpha_3}{2} + 8 \cdot \frac{\alpha_2 + \alpha_3}{2} \\ &= (\alpha_1 + \alpha_2 + \alpha_3) + 4(\alpha_1 + \alpha_2) + 4(\alpha_1 + \alpha_3) + 4(\alpha_2 + \alpha_3) \\ &= 9(\alpha_1 + \alpha_2 + \alpha_3) = 9\mu_0. \end{aligned}$$

Dividing by $\dim J_3(\mathbb{O}) = 27$ gives the second identity. $\square$

Remark 2.2. The spectral trace $9\mu_0 \approx 1233$ differs from the Jordan trace $\mu_0 \approx 137$ by a factor of 9. The Jordan trace is the rank-3 trace (sum of eigenvalues of X_{sec} as a 3×3 matrix); the spectral trace is the full-27-dimensional operator trace. Neither equals μ_0 alone, though their ratio $\text{Tr}_{\text{spec}}/9 = \text{Tr}_{J_3(\mathbb{O})}$ relates them precisely.

2.2 Gauss quadrature nodes: comparison with Peirce eigenvalues

The Gauss quadrature nodes for the weight $\rho(x)/\mu_0$ on $[0, 1]$ are computed by the Stieltjes–Gram–Schmidt algorithm applied to the moment sequence. The leading nodes are:

3-node: 0.33521, 0.68662, 0.93548.

The Peirce eigenvalues normalized by μ_0 are $\pi/\mu_0 \approx 0.0229$, $\pi^2/\mu_0 \approx 0.0720$, $4\pi^3/\mu_0 \approx 0.9051$. The 3-node Gauss points do not match these (nor do they match the Peirce center eigenvalues $2/3, 3/4, 4/5$).

Proposition 2.3 (No spectral identification). $\rho(x)$ is not the spectral density of $L_{X_{\text{sec}}}$:

- The supports are disjoint: $\text{supp}(\rho) = [0, 1]$, $\text{spec}(L_{X_{\text{sec}}}) \subset [\pi, 4\pi^3]$.
- Gauss quadrature nodes of ρ do not match Peirce eigenvalues under normalization by μ_0 , $4\pi^3$, or $\log(4\pi^3)$.
- The spectral k -th moment $\sum_i w_i \lambda_i^k$ grows as $(4\pi^3)^k$; the density moment $\mu_k \rightarrow 0$ as $k \rightarrow \infty$.

3 The sector trace generating function

3.1 Decomposition of ρ

From the sector construction (Addendum 36), the coefficient of x^{n_i} in ρ is $c_i = \alpha_i(n_i + 1)$:

$$\rho(x) = \sum_{i=1}^3 \alpha_i (n_i + 1) x^{n_i} = 2\pi x + 3\pi^2 x^2 + 16\pi^3 x^3. \quad (5)$$

(Check: $c_3 = 4\pi^3 \cdot 4 = 16\pi^3$.) Each normalized sector density $\rho_i^{(n)}(x) = (n_i + 1) x^{n_i}$ integrates to 1:

$$\rho(x) = \sum_i \alpha_i \rho_i^{(n)}(x), \quad \int_0^1 \rho_i^{(n)}(x) dx = 1.$$

3.2 Harmonic diagonal

Definition 3.1 (Harmonic diagonal). For $k \geq 0$, the *harmonic diagonal of order k* is

$$D_k = \text{diag}\left(\frac{2}{k+2}, \frac{3}{k+3}, \frac{4}{k+4}\right) = \text{diag}\left(\frac{n_1+1}{k+n_1+1}, \frac{n_2+1}{k+n_2+1}, \frac{n_3+1}{k+n_3+1}\right) \in V_1. \quad (6)$$

Properties: $D_0 = I_{V_1}$; $D_1 = K = \text{diag}(2/3, 3/4, 4/5)$ (the Peirce center matrix of Addendum 46); $D_k \rightarrow 0$ monotonically as $k \rightarrow \infty$; $(D_k)_i = 1/(1 + k/(n_i + 1))$ (harmonic damping).

Lemma 3.2 (Integral representation). $(D_k)_i = \int_0^1 x^k \rho_i^{(n)}(x) dx = (n_i + 1)/(k + n_i + 1)$.

Proof. Direct: $\int_0^1 x^k (n_i + 1) x^{n_i} dx = (n_i + 1)/(k + n_i + 1)$. □

3.3 Main theorem

Theorem 3.3 (Moment hierarchy as $J_3(\mathbb{O})$ trace hierarchy). *For all $k \geq 0$:*

$$\mu_k = \int_0^1 x^k \rho(x) dx = \text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}} \circ D_k). \quad (7)$$

Proof. Using (5) and Lemma 3.2:

$$\int_0^1 x^k \rho(x) dx = \sum_{i=1}^3 \alpha_i \int_0^1 x^k \rho_i^{(n)}(x) dx = \sum_{i=1}^3 \alpha_i (D_k)_i = \text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}} \circ D_k). \quad \square$$

□

Corollary 3.4. $\mu_0 = \text{Tr}(X_{\text{sec}} \circ D_0) = \text{Tr}(X_{\text{sec}}) = \mu_0$; $\mu_1 = \text{Tr}(X_{\text{sec}} \circ D_1) = \text{Tr}(X_{\text{sec}} \circ K) = \mu_1$. The Peirce center $K = D_1$ is the harmonic diagonal at $k = 1$.

4 The generating function as a Stieltjes series

Proposition 4.1 (Rational generating function). *The function $F(z) = \text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}} \circ D_z)$, defined for $z \notin \{-2, -3, -4\}$, is*

$$F(z) = \frac{2\pi}{z+2} + \frac{3\pi^2}{z+3} + \frac{16\pi^3}{z+4}, \quad (8)$$

with: (i) $\mu_k = F(k)$ for all $k \geq 0$; (ii) poles at $z = -(n_i + 1) = -2, -3, -4$ with residues $\alpha_i(n_i + 1) = 2\pi, 3\pi^2, 16\pi^3$; (iii) $F(0) = \mu_0$, $F(1) = \mu_1$, $F(z) \rightarrow 0$ as $z \rightarrow +\infty$.

Table 2: Harmonic diagonal D_k and corresponding moments. Entries verified numerically to 10^{-10} precision.

k	D_k	$\text{Tr}(X_{\text{sec}} \circ D_k)$	μ_k
0	$\text{diag}(1, 1, 1)$	137.036 304	137.036 304
1	$\text{diag}(\frac{2}{3}, \frac{3}{4}, \frac{4}{5})$	108.716 684	108.716 684
2	$\text{diag}(\frac{1}{2}, \frac{3}{5}, \frac{2}{3})$	90.175 963	90.175 963
3	$\text{diag}(\frac{2}{5}, \frac{1}{2}, \frac{4}{7})$	77.062 929	77.062 929
4	$\text{diag}(\frac{1}{3}, \frac{3}{7}, \frac{1}{2})$	67.289 581	67.289 581

Proof. $F(z) = \sum_i \alpha_i(n_i + 1)/(z + n_i + 1)$ by Definition 3.1. Substituting $n_i + 1 \in \{2, 3, 4\}$ and $\alpha_i(n_i + 1) \in \{2\pi, 3\pi^2, 16\pi^3\}$ gives (8). Items (i)–(iii) are direct. $\square$

Remark 4.2 (Poles and sector structure). The poles at $z = -2, -3, -4$ encode the sector monomial degrees n_i : each sector density $\rho_i^{(n)} \propto x^{n_i}$ integrates against x^z as $(z + n_i + 1)^{-1}$, producing a simple pole at $z = -(n_i + 1)$. The residues $\alpha_i(n_i + 1)$ are the sector energies weighted by the sector dimension coefficient. This is the exact Stieltjes structure of the three-sector V_1 decomposition.

Remark 4.3 (G_0 – G_1 as Stieltjes ratio). $\text{MU} = F(1)/F(0) = \mu_1/\mu_0$. The G_0 – G_1 gap of Addendum 46 is:

$$G_0 - G_1 = \frac{\ln F(0) - \ln F(1)}{\text{MU}} = \frac{-\ln \text{MU}}{\text{MU}} = \frac{-\ln(F(1)/F(0))}{F(1)/F(0)},$$

a function of consecutive Stieltjes values. Identifying $-\ln \text{MU}/\text{MU}$ as a $J_3(\mathbb{O})$ intrinsic invariant remains open (OP-D).

5 Consequences for Conjectures B and D

5.1 What is established

Theorem 5.1 (Route 3 summary). *The following exact identities hold, all following from Theorem 3.3 and Proposition 2.1:*

$$\mu_k = \text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}} \circ D_k) \quad \forall k \geq 0, \quad (9)$$

$$\mu_0 = \text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}}) = \mu_0, \quad (10)$$

$$\mu_1 = \text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}} \circ K) = \mu_1, \quad (11)$$

$$\frac{\text{Tr}_{\text{spec}}(L_{X_{\text{sec}}})}{\dim J_3(\mathbb{O})} = \frac{\mu_0}{3}. \quad (12)$$

The Peirce center matrix $K = D_1$ is the $k = 1$ case of the universal harmonic family D_k .

5.2 What is not established

Proposition 5.2 (Tautology and surviving gap). (a) Identity (10) is tautological: ρ was defined so that $\int_0^1 \rho dx = \alpha_1 + \alpha_2 + \alpha_3$, which by construction equals both μ_0 and $\text{Tr}(X_{\text{sec}})$.

(b) Conjecture B requires deriving $m_t/m_c = \text{Tr}(X_{\text{sec}})$ without invoking the mass formula. This requires identifying X_{sec} as the canonical $J_3(\mathbb{O})$ generation mass element from G_2 -orbit structure, not by postulation. Route 3 provides no new input for this.

(c) The spectral density identification fails (Proposition 2.3); the tautological trace identity survives but does not close OP-B.

5.3 Updated open problem status

OP-B (surviving) Derive $m_t/m_c = \text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}}) = \mu_0$ without using the mass formula in the forward direction. The exact $J_3(\mathbb{O})$ algebraic form is identified. Missing: a G_2 -orbit derivation of X_{sec} as canonical.

OP-D (surviving, reduces to OP-B) Given OP-B, Conjecture D is structural iff $-\ln \text{MU}/\text{MU} = -\ln(F(1)/F(0))/(F(1)/F(0))$ admits a $J_3(\mathbb{O})$ -intrinsic derivation. The Stieltjes form of F provides additional structure but no identification of $-\ln \text{MU}/\text{MU}$ as a Casimir or Killing invariant.

6 Summary

1. $\rho(x)$ is not the spectral density of $L_{X_{\text{sec}}}$ (Proposition 2.3). Route 3 as stated in Addendum 46 does not hold.
2. $\rho(x)$ is the sector trace generating polynomial (Theorem 3.3): $\mu_k = \text{Tr}_{J_3(\mathbb{O})}(X_{\text{sec}} \circ D_k)$ for all $k \geq 0$, with harmonic diagonal $D_k = \text{diag}(2/(k+2), 3/(k+3), 4/(k+4))$.
3. The Peirce center $K = D_1$ is the $k = 1$ harmonic diagonal, extending the G_0 – G_1 link of Addendum 46 to a full rational generating function $F(z)$ with poles at $z = -2, -3, -4$.
4. Spectral mean identity: $\text{Tr}_{\text{spec}}(L_{X_{\text{sec}}})/27 = \mu_0/3$ exactly (Proposition 2.1).
5. Conjectures B and D are not closed. The trace identity $\mu_0 = \text{Tr}(X_{\text{sec}})$ is tautological. OP-B survives unchanged.

Table 3: Status of Conjectures B and D after Addenda 46–47.

	Conjecture B	Conjecture D
$J_3(\mathbb{O})$ form	$m_t/m_c = \text{Tr}(X_{\text{sec}} \circ D_0) = \text{Tr}(X_{\text{sec}})$	$E_d = \ln(\text{Tr}(X_{\text{sec}} \circ D_1))/\text{MU}$
Exact identity	$\text{Tr}(X_{\text{sec}} \circ D_0) = \mu_0$ (tautological)	$\text{Tr}(X_{\text{sec}} \circ D_1) = \mu_1$ (tautological)
Route 3 verdict	Spectral identification fails; generating function holds	Same
New structure	Harmonic family D_k ; rational $F(z)$	$\text{MU} = F(1)/F(0)$ Stieltjes ratio
Remaining input	X_{sec} as G_2 -orbit canonical (not postulated)	$-\ln \text{MU}/\text{MU}$ as $J_3(\mathbb{O})$ invariant
Status	OP-B open	OP-D open (reduces to OP-B)

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