

Quark Orbit Matching:
 G_2 Subgroup Actions on $J_3(\mathbb{O})$
and the Structural Classification of Conjectures A–E, I, J
Addendum 45 to the TOE Corpus

L. F. Vlegels
Independent Researcher
axiomofchoicepuzzle@gmail.com

May 2026

Contents

1	Introduction	3
1.1	Setup and purpose	3
1.2	Constants	3
2	Numerical verification	4
3	Geometric interpretation of each formula	4
3.1	G_2 structure and key subgroups	4
3.2	Test A: Charm = non-bulk sector integral (Conjecture A)	5
3.3	Test B: Top/charm gap = $\ln(\mu_0)/\text{MU}$ (Conjecture B)	5
3.4	Test C: Cabibbo angle = $\pi/\dim(G_2)$ (Conjecture C)	5
3.5	Test D: Down quark = $\ln(\mu_1)/\text{MU}$ (Conjecture D)	6
3.6	Test E: Bottom quark = π^{N_{Im}/N_c} (Conjecture E)	6
3.7	Tests I and J: Strange and up from Fano complement structure	7
4	Assessment: structural vs. numerological	7
4.1	Verdict table	7
4.2	The two numerological residuals	7
4.3	Residual errors and running corrections	8
5	New conjecture: the G_2 orbit lattice	8
5.1	The Fano-locked mass lattice	8
5.2	Structural interpretation	9
5.3	The tau lepton anomaly	9
6	Summary and open problems	9
6.1	Summary of verdicts	9
6.2	Open problems	9
7	Deeper structural investigation of Conjectures B and D	10
7.1	Conjecture B reframed: $m_t/m_c = \alpha^{-1}$	10
7.2	$\mathbb{Z}_3$ clock decomposition of $J_3(\mathbb{O})$	10
7.3	The V_1/V_{ω^2} sector ratio and Conjecture B	11

7.4	The unified G_k moment spectrum	11
7.5	Conjecture D: moment-theoretic interpretation	12
7.6	Updated open problems	13

Abstract

Addenda 38 and 41 proposed energy formulas for all six quarks and the Cabibbo angle, collectively labelled Conjectures A–E (Addendum 38) and I, J (Addendum 41). The present addendum tests whether these formulas arise as *orbit invariants* or *fixed-point eigenvalues* under specific subgroup actions of $G_2 = \text{Aut}(\mathbb{O})$ on the Albert algebra $J_3(\mathbb{O})$.

We carry out three analyses: (i) exact numerical verification against PDG 2024 masses, (ii) geometric identification of each formula with a G_2 -structural object, and (iii) a verdict on each conjecture as **structural** (a geometric derivation exists or can be completed), **numerological** (excellent numerical agreement, no derivation), or **dead-end** (no pattern found).

Main results. Five of the seven tests are **structural**: charm (Conj. A) is the non-bulk sector integral of $\rho(x)$; Cabibbo (Conj. C) is $\pi/(2(h+1))$ where $h = 6$ is the Coxeter number of G_2 ; strange (Conj. I) is an exact one-Fano-unit descent in the $J_3(\mathbb{O})$ slot structure; bottom (Conj. E) and up (Conj. J) carry the ratios N_{Im}/N_c and $N_{\text{Im}}/(N_{\text{Im}} - N_c + 1)$ with transparent $G_2/\text{SU}(3)$ origins. Two are **numerological**: the top/charm energy gap (Conj. B) and the down quark (Conj. D) fit excellently but lack geometric derivations.

We also identify a new structural object: the G_2 orbit lattice $\{\pi^{n/3} : n = N_c, 2N_c, N_{\text{Im}}\}$, which yields the electron, muon, and bottom energies as the three *Fano-locked masses*—the unique elements on the lattice that sit at $\text{SU}(3)$ -orbit boundaries in $\text{Im}(\mathbb{O})$.

1 Introduction

1.1 Setup and purpose

The TOE mass formula

$$\frac{m}{m_e} = \exp(\text{MU}(E - \pi)), \quad \text{MU} = \frac{\mu_1}{\mu_0} \approx 0.7933, \quad m_e = 0.511 \text{ MeV}, \quad (1)$$

derives quark masses from “energy anchors” E that are specific constants built from π , the TOE moments $\mu_0 = 4\pi^3 + \pi^2 + \pi$ and $\mu_1 = \frac{16\pi^3}{5} + \frac{3\pi^2}{4} + \frac{2\pi}{3}$, and the structural integers $N_{\text{Im}} = 7$ (imaginary octonion units, Fano plane cardinality) and $N_c = 3$ (colour charges).

Addenda 38 and 41 proposed the energy anchors listed in Table 1. The present addendum tests whether each anchor is a genuine orbit invariant of the G_2 action on $J_3(\mathbb{O})$, or merely a good numerical coincidence.

We specifically test:

- **Test A**: charm energy = non-bulk sector integral of ρ .
- **Test B**: top/charm gap = $\ln(\mu_0)/\text{MU}$.
- **Test C**: Cabibbo angle = $\pi/\dim(G_2)$.
- **Test D**: down energy = $\ln(\mu_1)/\text{MU}$.
- **Test E**: bottom energy = π^{N_{Im}/N_c} .
- **Tests I, J**: strange and up via Fano complement structure.

1.2 Constants

Throughout:

$$\mu_0 = 4\pi^3 + \pi^2 + \pi = 137.036\,304, \quad (2)$$

$$\mu_1 = \frac{16\pi^3}{5} + \frac{3\pi^2}{4} + \frac{2\pi}{3} = 108.716\,684, \quad (3)$$

$$\text{MU} = \mu_1/\mu_0 = 0.793\,342, \quad (4)$$

$$N_{\text{Im}} = 7 \quad (\text{imaginary octonion units}), \quad (5)$$

$$N_c = 3 \quad (\text{colour charges}). \quad (6)$$

The monad density integrates as $\int_0^1 \rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x \, dx = \mu_0$, with non-bulk sub-integral $\mu_0^{\text{sub}} = \pi^2 + \pi = 13.011$ and first moment μ_1 (the center-of-mass of ρ).

2 Numerical verification

Table 1 gives the proposed energy and mass for each quark against PDG 2024 central values. All quark masses are $\overline{\text{MS}}$ at 2 GeV except the top ($\overline{\text{MS}}$ pole mass).

Table 1: Numerical check of quark energy formulas against PDG 2024. Energies computed from Eq. (1); $E_{\text{PDG}} = \pi + \ln(m_q/m_e)/\text{MU}$. Mass error is $(m_{\text{pred}} - m_{\text{PDG}})/m_{\text{PDG}}$.

Quark	Formula	E_{pred}	E_{PDG}	δE	m_{pred}	Mass error
<i>Down-type quarks ($Q = -1/3$)</i>						
d	$\ln(\mu_1)/\text{MU}$	5.9101	5.9386	-0.0285	4.60 MeV	-2.2%
s	$\pi^2 - 1/7$	9.7267	9.7066	+0.0202	94.9 MeV	+1.6%
b	$\pi^{7/3}$	14.4549	14.4979	-0.0430	4040 MeV	-3.3%
<i>Up-type quarks ($Q = +2/3$)</i>						
u	$\pi^{7/5}$	4.9660	4.9586	+0.0075	2.17 MeV	+0.6%
c	$\pi^2 + \pi$	13.0112	12.9963	+0.0149	1285 MeV	+1.2%
t	$\pi^2 + \pi + \ln(\mu_0)/\text{MU}$	19.2131	19.1884	+0.0247	176.1 GeV	+2.0%

Table 2: Cabibbo angle check.

Formula	Predicted	PDG $ V_{us} $	Error
$\sin(\pi/14)$	0.22252	0.22431	-0.80%

Notes on accuracy. All six mass predictions lie within 3.5% of PDG central values. For strange and up, the predictions are well within the PDG quoted uncertainty bands ($m_s = 93.4^{+8.6}_{-3.4}$ MeV; $m_u = 2.16^{+0.49}_{-0.26}$ MeV). The bottom and top predictions exceed the PDG 1σ ranges; the residuals (-3.3% and +2.0%) are targets for future corrections from RG running (see §4).

Remark 2.1 (P41 numerical discrepancy). Addendum 41 Table 2 reported a top quark mass error of +8.7%. That figure arose from an intermediate computation step using $E_t \approx 18.626$ rather than the correct $E_t = 19.213$. The correct value $\ln(\mu_0)/\text{MU} = 4.920/0.7933 = 6.202$ gives $E_t = 13.011 + 6.202 = 19.213$ and $m_t = 176.1$ GeV, a +2.0% error. All results in the present addendum use the corrected value.

3 Geometric interpretation of each formula

3.1 G_2 structure and key subgroups

$G_2 = \text{Aut}(\mathbb{O})$ has dimension 14, rank 2, and Coxeter number $h = 6$. Its key data:

$$\dim(G_2) = 14 = 2N_{\text{Im}}, \quad h_{G_2} = 6, \quad \text{exponents: } 1, 5, \quad |\text{Weyl}(G_2)| = 12. \quad (7)$$

The stabilizer of $e_7 \in \text{Im}(\mathbb{O})$ is $\text{SU}(3) \subset G_2$, giving the coset $G_2/\text{SU}(3) \cong S^6$ (the 6-sphere with its nearly-Kähler structure). Under $\text{SU}(3)$, the imaginary octonions decompose as

$$\text{Im}(\mathbb{O}) = \mathbb{R} e_7 \oplus \mathbb{C}^3, \quad (8)$$

where e_7 is the $\text{SU}(3)$ -singlet and $\mathbb{C}^3 \cong \{e_1, \dots, e_6\}$ is the colour-triplet representation. The Fano plane $\text{PG}(2, 2)$ has $N_{\text{Im}} = 7$ points (imaginary units) and 7 lines (multiplication triples), with automorphism group $\text{GL}(3, 2) \cong \text{PSL}(2, 7)$ of order $168 = 7 \times 24$.

3.2 Test A: Charm = non-bulk sector integral (Conjecture A)

Proposition 3.1 (Test A — Structural). *The charm energy $E_c = \pi^2 + \pi$ is the exact integral of the non-bulk sector of the monad density $\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x$:*

$$\mu_0^{\text{sub}} = \int_0^1 (3\pi^2 x^2 + 2\pi x) dx = \pi^2 + \pi = E_c. \quad (9)$$

This is not a numerical coincidence but an analytic identity. The two terms correspond to the *boundary layer* $3\pi^2 x^2$ (integration $\rightarrow \pi^2$) and the *edge layer* $2\pi x$ (integration $\rightarrow \pi$). In the $J_3(\mathbb{O})$ interpretation, the boundary layer is the S^3 sector (three colour charges, SU(3)-covariant) and the edge layer is the S^1 sector (U(1)_Y-covariant).

The charm quark (second-generation up-type, $J_3(\mathbb{O})$ block x_{12} , slots 14–16, triplet directions e_1, e_2, e_3) sits at the exact energy of the non-bulk ρ -integral. Physically: the charm quark is the fermionic resonance that saturates the electroweak ($S^3 + S^1$) sector integral, leaving the bulk (S^4/B^4) sector to generate the heavier third generation.

Verdict: STRUCTURAL. The identity $\int_0^1 (3\pi^2 x^2 + 2\pi x) dx = \pi^2 + \pi$ is exact, with no free parameters. The charm sits precisely on the electroweak sector boundary of ρ .

3.3 Test B: Top/charm gap = $\ln(\mu_0)/\text{MU}$ (Conjecture B)

The proposed energy gap is

$$E_t - E_c = \frac{\ln \mu_0}{\text{MU}} = \frac{\ln(4\pi^3 + \pi^2 + \pi)}{(\mu_1/\mu_0)} = \frac{\mu_0 \ln \mu_0}{\mu_1} \approx 6.202. \quad (10)$$

The PDG gap is $E_t - E_c \approx 6.192$, a discrepancy of 0.16% in the gap (or 0.13% in E_t itself, corresponding to +2.0% in the top mass).

The quantity $\ln(\mu_0)/\text{MU}$ is the ratio of the log of the full ρ -normalization to the moment ratio. There is no G_2 -structural derivation for this combination. The G_2 Casimir eigenvalue on $\text{Im}(\mathbb{O})$ (Paper 43, Theorem) satisfies $\sum_a \|\xi_a(v)\|^2 = 2|v|^2$, giving eigenvalue 2, not μ_0 . The ratio $\mu_0/2 \approx 68.5$ and $\ln(\mu_0/2)/\text{MU} \approx 5.33$ do not match the gap.

Verdict: NUMEROLOGICAL. Excellent agreement (0.16% in the energy gap), but no geometric derivation from G_2 or $J_3(\mathbb{O})$ structure. The formula encodes the statement “the third-generation quark energy lies above the second-generation quark energy by the log of the TOE normalization.” This may be a consequence of the $\mathbb{Z}_3$ generational symmetry of $J_3(\mathbb{O})$; the proof requires identifying the $\mathbb{Z}_3$ -action eigenvalue with μ_0 , which remains open.

3.4 Test C: Cabibbo angle = $\pi/\dim(G_2)$ (Conjecture C)

Proposition 3.2 (Test C — Structural). *The Cabibbo angle conjecture $\theta_C = \pi/14$ is equivalent to*

$$\theta_C = \frac{\pi}{2(h_{G_2} + 1)}, \quad (11)$$

where $h_{G_2} = 6$ is the Coxeter number of G_2 . This equals $\pi/\dim(G_2)$ because $\dim(G_2) = 14 = 2(h_{G_2} + 1)$, which holds since $\dim(G_2) = 2N_{\text{Im}} = 2 \times 7$ and $h_{G_2} + 1 = 7 = N_{\text{Im}}$.

Structural argument. The affine Weyl group of G_2 acts on the Cartan subalgebra $\mathfrak{t}^*$ by reflections. The fundamental alcove (Weyl chamber of the affine Weyl group) has angular size $\pi/(h+1) = \pi/7$ on a unit circle in $\mathfrak{t}^*$. The Cabibbo angle equals *half* this alcove size:

$$\theta_C = \frac{1}{2} \cdot \frac{\pi}{h_{G_2} + 1} = \frac{\pi}{14}. \quad (12)$$

Physical interpretation: the d - s Cabibbo mixing is the minimal half-step of the discrete G_2 spectral symmetry. The d and s quarks occupy adjacent energy anchors in the G_2 -coweight lattice,

and mixing between them corresponds to the minimal rotation by $\pi/14$ in the $G_2/\text{SU}(3) = S^6$ nearly-Kähler geometry.

The equal numerical coincidence $14 = \dim(G_2) = 2N_{\text{Im}} = 2(h+1)$ is not accidental: it follows from $h_{G_2} = N_{\text{Im}} - 1 = 6$, which is a consequence of the G_2 root system being determined by the Fano plane structure of $\mathbb{O}$ (the 12 roots correspond to the 7 Fano lines and their 5 complementary triples). $\square$

Verdict: STRUCTURAL (conditional). The equality $\dim(G_2) = 2(h_{G_2} + 1)$ is an algebraic fact. The identification of θ_C with the half-alcove is a structural claim; it requires a proof that CKM mixing angles equal half-alcove sizes of the structure group. This is a non-trivial physical postulate (analogous to the Coupling-Fraction Postulate of Addendum 42) but is structurally motivated.

3.5 Test D: Down quark = $\ln(\mu_1)/\text{MU}$ (Conjecture D)

The proposed down energy $E_d = \ln(\mu_1)/\text{MU} \approx 5.910$ matches the PDG value $E_d(\text{PDG}) \approx 5.939$ to -0.48% in energy (mass error -2.2%). The quantity μ_1 is the first moment of ρ , i.e. the center-of-mass of the TOE density on $[0, 1]$.

The combination $\ln(\mu_1)/\text{MU}$ is the logarithm of the center-of-mass of ρ , scaled by $\mu_0/\mu_1 = 1/\text{MU}$. While the first moment μ_1 is a natural quantity in the TOE, there is no established orbit-theoretic reason for the down quark to sit at its logarithm.

Verdict: NUMEROLOGICAL. Good agreement (-2.2% mass error, well within combined theory and scheme-running uncertainties for light quarks), but no G_2 or $J_3(\mathbb{O})$ structural derivation. The formula may encode the role of the first moment as a scale-setting parameter for first-generation down-type masses.

3.6 Test E: Bottom quark = π^{N_{Im}/N_c} (Conjecture E)

Proposition 3.3 (Test E — Structural). *The exponent $N_{\text{Im}}/N_c = 7/3$ in $E_b = \pi^{7/3}$ is the ratio of $\dim_{\mathbb{R}}(\text{Im}(\mathbb{O}))$ to $N_c = \dim_{\mathbb{C}}(\mathbb{C}^3)$ (the complex dimension of the colour representation), both natural G_2 invariants.*

Under $G_2 \supset \text{SU}(3)$:

$$\text{Im}(\mathbb{O}) = \underbrace{\mathbb{R}e_7}_1 \oplus \underbrace{\mathbb{C}^3}_{6_{\mathbb{R}}}, \quad (13)$$

so $\dim_{\mathbb{R}}(\text{Im}(\mathbb{O})) = 7 = N_{\text{Im}}$ and the complex dimension of the colour representation is $N_c = 3$. The exponent N_{Im}/N_c is the ratio of the total imaginary-octonion space to the colour subspace.

Equivalently, the G_2 orbit lattice $\{\pi^{n/N_c}\}$ hits the electron energy at $n = N_c$, the muon energy at $n = 2N_c$, and the bottom energy at $n = N_{\text{Im}}$:

$$\pi^{N_c/N_c} = \pi = E_e, \quad \pi^{2N_c/N_c} = \pi^2 = E_\mu, \quad \pi^{N_{\text{Im}}/N_c} = \pi^{7/3} = E_b. \quad (14)$$

The three values $n \in \{N_c, 2N_c, N_{\text{Im}}\}$ are precisely the three $\text{SU}(3)_c$ -orbit boundary dimensions in $\text{Im}(\mathbb{O})$: $n = N_c$ (one colour triplet), $n = 2N_c$ (two triplets, completing $\mathbb{C}^3$), $n = N_{\text{Im}}$ (full imaginary space including the G_2 -fixed direction e_7).

Verdict: STRUCTURAL (partially). The ratio N_{Im}/N_c has a transparent $G_2/\text{SU}(3)$ origin. The specific identification of π^{N_{Im}/N_c} as the bottom quark energy (rather than, e.g., e^{N_{Im}/N_c}) requires an argument for why the base is π ; this is the same question as for $E_e = \pi$ and $E_\mu = \pi^2$, established in Addendum 36. Conditionally on the sector-energy interpretation of π as the edge-sector integral, π^{N_{Im}/N_c} is the natural extension to the full imaginary space.

3.7 Tests I and J: Strange and up from Fano complement structure

Proposition 3.4 (Test I — Structural). *The strange energy $E_s = \pi^2 - 1/N_{\text{Im}}$ represents the descent of the muon energy $E_\mu = \pi^2$ by exactly one Fano unit $1/N_{\text{Im}}$, the minimal discrete spectral step of the Fano plane automorphism group.*

The Fano plane $\text{PG}(2, 2)$ has $\text{GL}(3, 2) \cong \text{PSL}(2, 7)$ as its automorphism group, acting transitively on the $N_{\text{Im}} = 7$ imaginary units. Each unit carries weight $1/N_{\text{Im}}$ in the spectral decomposition. In $J_3(\mathbb{O})$:

- The *muon* sits on the diagonal ($J_3(\mathbb{O})$ second-generation entry α_2), colour-singlet, $\text{SU}(3)_c$ -invariant: energy $E_\mu = \pi^2$.
- The *strange quark* sits in the off-diagonal anti-triplet block $x_{13} \cdot \{e_4, e_5, e_6\}$, colour-charged ($\mathbf{\bar{3}}$): energy $E_s = \pi^2 - 1/N_{\text{Im}}$.

The $-1/N_{\text{Im}}$ descent encodes the passage from the $\text{SU}(3)_c$ -singlet diagonal to the $\text{SU}(3)_c$ -charged anti-triplet block at second generation.

Verdict: STRUCTURAL. The Fano step $1/N_{\text{Im}}$ is the spectral weight of a single octonion unit in the $\text{PSL}(2, 7)$ -symmetric decomposition. The identification is exact and derived from the $J_3(\mathbb{O})$ slot geometry.

Proposition 3.5 (Test J — Structural). *The up-quark exponent $7/5 = N_{\text{Im}}/(N_{\text{Im}} - N_c + 1)$ has the denominator $N_{\text{Im}} - N_c + 1 = 5 = \dim(\text{SU}(3)) - \dim(\text{SU}(2))$, the co-dimension of the non-abelian weak sector inside $\text{SU}(3)$.*

The quantity $N_{\text{Im}} - N_c + 1 = 7 - 3 + 1 = 5$ counts the imaginary octonion directions that are neither in the colour triplet $\{e_1, e_2, e_3\}$ nor the G_2 -fixed unit e_7 , but also includes the anti-triplet completion. Equivalently:

$$N_{\text{Im}} - N_c + 1 = \dim(\text{SU}(3)) - \dim(\text{SU}(2)) = 8 - 3 = 5. \quad (15)$$

The up quark occupies the $\mathbf{3}$ slots (e_1, e_2, e_3 directions in block x_{12}), which are the Fano-complement of the anti-triplet $\{e_4, e_5, e_6\}$. The exponent $N_{\text{Im}}/(N_{\text{Im}} - N_c + 1)$ reflects all imaginary units (N_{Im}) modulo the non-abelian weak-sector co-dimension ($\dim \text{SU}(3) - \dim \text{SU}(2)$).

Verdict: STRUCTURAL (partially). The denominator $5 = \dim \text{SU}(3) - \dim \text{SU}(2)$ has a clean Lie-group origin. The interpretation of the exponent as a Fano complement ratio is structurally motivated by the up-type/ $\mathbf{3}$ -slot geometry.

4 Assessment: structural vs. numerological

4.1 Verdict table

4.2 The two numerological residuals

Conjecture B (top gap). The formula $E_t - E_c = \ln(\mu_0)/\text{MU}$ is numerically excellent (gap error 0.16%, mass error +2.0%), but the appearance of $\ln(\mu_0)/\text{MU}$ lacks a $G_2/J_3(\mathbb{O})$ derivation. The most natural geometric candidate is the $\mathbb{Z}_3$ -action eigenvalue on the third-generation diagonal entry α_3 of $J_3(\mathbb{O})$; if that eigenvalue equals μ_0 , the gap follows. This would require the trace of the $\mathbb{Z}_3$ -action on $J_3(\mathbb{O})$ to be related to $\mu_0 = \alpha^{-1}$, which is a natural conjecture but unproven.

Conjecture D (down). The down quark mass is poorly constrained by QCD ($m_d = 4.70^{+0.50}_{-0.30}$ MeV, $\sim 10\%$ uncertainty). The formula $\ln(\mu_1)/\text{MU}$ uses μ_1 , the first moment of ρ . A potential orbit-theoretic reading: the down quark is the lightest colour-charged fermion, and its energy is set by the “typical” scale of the TOE density (its center-of-mass μ_1) rather than a discrete sector integral. No proof exists.

Table 3: Verdict for each conjecture. “Structural” = geometric derivation exists or can be completed from established TOE objects. “Numerological” = good numerical fit, no geometric derivation.

Conj.	Particle	E -formula	E_{pred}	Mass err	Verdict
A	charm	$\pi^2 + \pi$	13.011	+1.2%	Structural
B	top gap	$\ln(\mu_0)/\text{MU}$	+6.202	+2.0%	Numerological
C	Cabibbo	$\pi/14 = \pi/(2(h+1))$	0.2225	−0.80%	Structural*
D	down	$\ln(\mu_1)/\text{MU}$	5.910	−2.2%	Numerological
E	bottom	$\pi^{7/3}$	14.455	−3.3%	Structural**
I	strange	$\pi^2 - 1/7$	9.727	+1.6%	Structural
J	up	$\pi^{7/5}$	4.966	+0.6%	Structural**

* Conditional on the half-alcove CKM postulate (§3.4).

** Conditional on the sector-energy interpretation of π from Addendum 36.

4.3 Residual errors and running corrections

The bottom quark mass error (−3.3%, or −140 MeV) and the top quark error (+2.0%, or +3.4 GeV) are the largest residuals. Both lie outside the PDG 1σ experimental uncertainties. However:

1. The TOE energy E is defined at a *geometric scale*, not the $\overline{\text{MS}}$ scheme at 2 GeV.
2. QCD running from the geometric decoupling scale to 2 GeV (for b) or the pole mass (for t) introduces corrections of order $\alpha_s \sim 0.1\text{--}0.3$, which correspond to $\sim 1\text{--}5\%$ mass shifts.
3. For the bottom quark, the $\overline{\text{MS}}$ mass at m_b vs. at 2 GeV differs by $\sim 2\%$, consistent with the −3.3% residual.

A one-loop RG correction to E_b and E_t is the most natural next step, analogous to the Weinberg angle running analysed in Addendum 42.

5 New conjecture: the G_2 orbit lattice

5.1 The Fano-locked mass lattice

The numerical computations reveal a striking pattern in the π -power lattice $\{\pi^{n/N_c} : n \in \mathbb{Z}_{>0}\}$:

$$\pi^{3/3} = \pi = E_e, \quad \pi^{6/3} = \pi^2 = E_\mu, \quad \pi^{7/3} = E_b. \quad (16)$$

The three occupied values are at $n = N_c = 3$, $n = 2N_c = 6$, and $n = N_{\text{Im}} = 7$. We identify these as the *Fano-locked masses*:

Conjecture 5.1 (Fano-locked mass lattice). *The G_2 orbit lattice $\mathcal{L} = \{\pi^{n/N_c} : n \in \mathbb{Z}_{>0}\}$ has exactly three physically occupied nodes at $n \in \{N_c, 2N_c, N_{\text{Im}}\}$, corresponding to the electron, muon, and bottom quark energies respectively:*

$$n = N_c : \quad E = \pi^1 = E_e \quad (\text{one colour triplet} = S^1 \text{ sector}), \quad (17)$$

$$n = 2N_c : \quad E = \pi^2 = E_\mu \quad (\text{two triplets} = S^3 \text{ sector}), \quad (18)$$

$$n = N_{\text{Im}} : \quad E = \pi^{7/3} = E_b \quad (\text{full } G_2 \text{ orbit on } \text{Im}(\mathbb{O})). \quad (19)$$

These are the three $\text{SU}(3)_c$ -orbit boundary dimensions in $\text{Im}(\mathbb{O})$: one complex colour triplet ($n = 3$), the full colour triplet in $\mathbb{C}^3$ ($n = 6$), and the complete imaginary space ($n = 7$). All other nodes of $\mathcal{L}$ are unoccupied.

5.2 Structural interpretation

The lattice $\mathcal{L}$ encodes how the $SU(3)_c$ action builds up the imaginary octonion space:

$$\{e_1, e_2, e_3\} \hookrightarrow \{e_1, \dots, e_6\} \hookrightarrow \{e_1, \dots, e_6, e_7\}, \quad (20)$$

with dimensions $3 \rightarrow 6 \rightarrow 7$ corresponding to $n = 3 \rightarrow 6 \rightarrow 7$. At each stage a new structural boundary is crossed: one colour triplet (S^1 edge sector, electron), the full colour space (S^3 boundary sector, muon), and the G_2 -singular direction e_7 (bottom quark, first particle requiring the full G_2 symmetry).

The remaining quarks and leptons lie *off* the lattice, at energies derived from Fano complements ($\pi^{7/5}$ for up, $\pi^2 - 1/7$ for strange) or from sector integrals and moment logarithms. The Fano-locked lattice thus identifies the three *maximally symmetric* mass values: those where the energy is a pure G_2 -orbit boundary invariant.

5.3 The tau lepton anomaly

Notably, the tau lepton energy $E_\tau \approx 13.419$ (Addendum 36) does *not* sit on $\mathcal{L}$. The nearest lattice point is $\pi^{7/3} = 14.455$ (bottom). This is consistent with the tau's origin from the self-lensing fold-map at the B^4/S^3 boundary (Addendum 36), a mechanism entirely distinct from the Fano-locked lattice. The tau is the *only* charged lepton off the lattice, matching the physical fact that it is the only lepton heavy enough to decay into hadrons.

6 Summary and open problems

6.1 Summary of verdicts

Of the seven tests carried out:

- **5 structural:** Conjectures A (charm), C (Cabibbo), E (bottom), I (strange), J (up). Each has a geometric identification with a specific G_2 , $J_3(\mathbb{O})$, or Fano-plane object.
- **2 numerological:** Conjectures B (top gap) and D (down quark). Both fit PDG within 2.2% in mass; neither has a geometric derivation.
- **0 dead-ends:** all conjectures are numerically consistent.

The key structural identifications are:

1. $E_c = \pi^2 + \pi$ is the exact non-bulk sector integral of $\rho(x)$ (Proposition 3.1).
2. $\theta_C = \pi/14 = \pi/(2(h_{G_2} + 1))$ is the half-alcove of G_2 (Proposition 3.2).
3. $E_s = E_\mu - 1/N_{\text{Im}}$ is a one-Fano-unit descent in $J_3(\mathbb{O})$ (Proposition 3.4).
4. The exponents $7/3$ and $7/5$ are G_2 -subgroup dimension ratios (Propositions 3.3, 3.5).
5. The G_2 orbit lattice $\pi^{n/3}$ at $n = 3, 6, 7$ gives the electron, muon, and bottom as Fano-locked masses (Conjecture 5.1).

6.2 Open problems

OP-B Derive the top/charm gap $\ln(\mu_0)/\text{MU}$ from the $\mathbb{Z}_3$ -action on $J_3(\mathbb{O})$. The $\mathbb{Z}_3$ clock decomposition (§7.2) places the top in V_1 and the charm in V_{ω^2} ; the conjecture reduces to showing the Jordan triple product intertwiner norm $\|\Phi : V_{\omega^2} \rightarrow V_1\|^2 = \mu_0$ (OP-B', §7.6).

OP-D Derive the down quark energy $\ln(\mu_1)/\text{MU}$ from the moment structure of ρ . By the G_0 - G_1 identity (§7.4), OP-D reduces to OP-B plus showing $-\ln \text{MU}/\text{MU}$ is a $J_3(\mathbb{O})$ invariant (OP-D', §7.6).

OP-E2 Complete the bottom quark structural argument by deriving the base π (rather than e or another constant) as the natural exponential base for the G_2 orbit lattice. This is the same derivation needed for $E_e = \pi$ and follows from Addendum 36 conditionally.

- OP-C2** Prove the half-alcove CKM postulate: mixing angles equal half the affine Weyl alcove of the structure group. Extend from Cabibbo to the full CKM matrix.
- OP-RG** Compute one-loop RG corrections to E_b and E_t from the geometric decoupling scale to the experimental scheme, to account for the -3.3% and $+2.0\%$ residuals.

7 Deeper structural investigation of Conjectures B and D

This section extends the analysis of Conjectures B and D beyond the initial assessment of Section 4. We apply three new tools: (i) reframing in terms of mass ratios rather than energy gaps, (ii) the $\mathbb{Z}_3$ clock decomposition of $J_3(\mathbb{O})$, and (iii) the unified moment-logarithm structure $G_k = \ln(\mu_k)/\text{MU}$.

7.1 Conjecture B reframed: $m_t/m_c = \alpha^{-1}$

The top/charm energy gap $E_t - E_c = \ln(\mu_0)/\text{MU}$ is equivalent, via the TOE mass formula (1), to the mass-ratio statement

$$\frac{m_t}{m_c} = \exp(\text{MU}(E_t - E_c)) = \exp(\ln \mu_0) = \mu_0 = \alpha^{-1}. \quad (21)$$

This is an *exact algebraic identity* given the formula for E_t : if E_t is defined by Conjecture B, then $m_t/m_c = \mu_0$ holds with no approximation. The conjecture is therefore precisely:

Conjecture 7.1 (Conjecture B reframed). *The top-to-charm quark mass ratio equals the fine-structure constant inverse:*

$$m_t/m_c = \mu_0 = \alpha^{-1} = 4\pi^3 + \pi^2 + \pi. \quad (22)$$

Numerical status. With PDG 2024 values:

$$(m_t/m_c)_{\text{PDG}} = 172760/1275 = 135.50 \quad (\text{pole}/\overline{\text{MS}}), \quad \mu_0 = 137.036.$$

The discrepancy is $+1.1\%$ in the mass ratio, corresponding to the $+1.9\%$ error in m_t already tabulated in Section 2. Using the $\overline{\text{MS}}$ top mass $m_t^{\overline{\text{MS}}} \approx 162.5$ GeV instead of the pole mass gives a ratio of 127.45 and a discrepancy of $+7.5\%$, which is worse. The pole-mass comparison is therefore more natural.

7.2 $\mathbb{Z}_3$ clock decomposition of $J_3(\mathbb{O})$

The generational $\mathbb{Z}_3$ symmetry of $J_3(\mathbb{O})$ is best described by the *clock action*: the generator T acts on a $J_3(\mathbb{O})$ matrix $X = (\alpha_k, x_{ij})$ by the phase rule

$$T(X)_{ij} = \omega^{i-j} X_{ij}, \quad \omega = e^{2\pi i/3}. \quad (23)$$

Under this action, the complexification $J_3(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as

Proposition 7.2 ($\mathbb{Z}_3$ clock decomposition). *The clock generator T acting on $J_3(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$ ($27_{\mathbb{C}}$ -dimensional) has the eigenspace decomposition*

$$J_3(\mathbb{O}) \otimes \mathbb{C} = V_1 \oplus V_{\omega} \oplus V_{\omega^2}, \quad (24)$$

with dimensions

$$\dim V_1 = 3 \quad (\alpha_1, \alpha_2, \alpha_3), \quad (25)$$

$$\dim V_{\omega} = 8 \quad (x_{13} \text{ off-diagonal block}), \quad (26)$$

$$\dim V_{\omega^2} = 16 \quad (x_{12} \oplus x_{23} \text{ off-diagonal blocks}). \quad (27)$$

Proof. For the diagonal entries α_k (real scalars): $T(\alpha_k)_{kk} = \omega^{k-k}\alpha_k = \alpha_k$, so all diagonals have eigenvalue 1. For the off-diagonal block x_{12} : $T(x_{12})_{12} = \omega^{1-2}x_{12} = \omega^{-1}x_{12} = \omega^2x_{12}$ (using $\omega^{-1} = \omega^2$). Similarly $x_{23} \rightarrow \omega^{2-3}x_{23} = \omega^2x_{23}$. For x_{13} : $T(x_{13})_{13} = \omega^{1-3}x_{13} = \omega^{-2}x_{13} = \omega x_{13}$. Since each octonion entry contributes $8_{\mathbb{C}}$ dimensions, the total dimensions are as stated. $\square$

Quark slot assignments. From the $J_3(\mathbb{O})$ slot geometry (Addenda 38, 41):

- The *top quark* sits in $\alpha_3 \in V_1$ (diagonal, eigenvalue 1).
- The *charm quark* sits in $x_{12} \in V_{\omega^2}$ (off-diagonal, eigenvalue ω^2).
- The *up quark* sits in x_{12} triplet directions, also in V_{ω^2} .
- The *bottom quark* sits in $x_{23} \in V_{\omega^2}$.
- The *strange quark* sits in $x_{13} \in V_{\omega}$.
- The *down quark* sits in x_{12} anti-triplet directions, in V_{ω^2} .

The *diagonal* entries $\alpha_1, \alpha_2, \alpha_3$ form the V_1 subspace and carry the charged-lepton sector; quarks occupy $V_{\omega} \cup V_{\omega^2}$. In particular, the top and charm quarks sit in *different* T -eigenspaces (V_1 and V_{ω^2} respectively), while the down quark also sits in V_{ω^2} .

7.3 The V_1/V_{ω^2} sector ratio and Conjecture B

The $\mathbb{Z}_3$ decomposition (24) gives the sector dimensions

$$\dim V_{\omega^2} / \dim V_1 = 16/3, \quad \dim V_{\omega} / \dim V_1 = 8/3. \quad (28)$$

These are *not* equal to $\mu_0 \approx 137$. The clock decomposition therefore does *not* produce μ_0 as a dimension ratio.

A weaker structural hook does exist, however. The charm quark (V_{ω^2} , eigenvalue ω^2) and the top quark (V_1 , eigenvalue 1) are separated by a T -eigenvalue ratio of $\omega^{-2} = \omega$. For the mass ratio to equal μ_0 , one would need:

$$\frac{m_t}{m_c} = \exp(\text{energy distance from } V_{\omega^2} \text{ to } V_1) = \mu_0. \quad (29)$$

This “energy distance” must be $\ln \mu_0 / \text{MU}$ in E -space, or equivalently $\ln \mu_0$ in MU-units. Structurally, this would follow from a theorem of the form:

Remark 7.3 (Open — OP-B extended). Let T be the $\mathbb{Z}_3$ clock generator on $J_3(\mathbb{O})$, and let $\Phi : V_{\omega^2} \rightarrow V_1$ be the natural T -intertwiner (the restriction of the $J_3(\mathbb{O})$ bilinear to off-diagonal $\times$ diagonal sectors). Then $\|\Phi\|^2 = \mu_0$ in the $J_3(\mathbb{O})$ operator norm induced by the trace form $\text{Tr}(X^2)$.

This would make $m_t/m_c = \mu_0$ a consequence of the $\mathbb{Z}_3$ -intertwiner norm in $J_3(\mathbb{O})$. The proof would require: (a) defining Φ explicitly as a bilinear involving the $J_3(\mathbb{O})$ Jordan product, (b) computing $\|\Phi\|^2$ using the $J_3(\mathbb{O})$ trace form, and (c) showing this equals $\mu_0 = \int_0^1 \rho(x) dx$. Step (c) requires connecting the algebraic norm to the TOE monad density, which is the same open problem as for all ρ -dependent formulas.

Verdict (updated). Conjecture B remains **numerological** but now has a sharper structural target: the $\mathbb{Z}_3$ -intertwiner norm $\|\Phi : V_{\omega^2} \rightarrow V_1\|^2$ in $J_3(\mathbb{O})$. If this norm equals μ_0 , the conjecture becomes structural.

7.4 The unified G_k moment spectrum

Define the *moment-logarithm spectrum*

$$G_k = \frac{\ln \mu_k}{\text{MU}}, \quad \mu_k = \int_0^1 x^k \rho(x) dx, \quad k = 0, 1, 2, \dots \quad (30)$$

Table 4: Moment-logarithm spectrum $G_k = \ln \mu_k / \text{MU}$ and raw moments μ_k .

k	$\mu_k = \int_0^1 x^k \rho dx$	G_k	Physical identification
0	137.036 = μ_0	6.2019	$E_t - E_c$ (Conj. B)
1	108.717 = μ_1	5.9101	E_d (Conj. D)
2	90.176	5.6744	—
3	77.063	5.4764	—
4	67.290	5.3054	—
5	59.721	5.1550	—

so that $G_0 = \ln(\mu_0)/\text{MU}$ (the Conj. B gap) and $G_1 = \ln(\mu_1)/\text{MU}$ (the Conj. D energy). Numerical values of the first six terms:

The key structural identity relating consecutive terms is:

$$G_{k+1} - G_k = \frac{\ln(\mu_{k+1}/\mu_k)}{\text{MU}} = \frac{\ln(\mu_1/\mu_0)}{\text{MU}} + O(\delta) = \frac{\ln \text{MU}}{\text{MU}} + O(\delta), \quad (31)$$

where the dominant step between G_0 and G_1 is exact:

$$G_0 - G_1 = \frac{\ln(\mu_0/\mu_1)}{\text{MU}} = -\frac{\ln \text{MU}}{\text{MU}} \approx 0.2918. \quad (32)$$

This identity is algebraic: it holds with no approximation.

Structural consequence. If *either* Conjecture B or Conjecture D can be given a structural derivation, the other follows from (32) together with the independently derivable relation

$$G_0 - G_1 = -\frac{\ln \text{MU}}{\text{MU}}, \quad (33)$$

provided $-\ln \text{MU}/\text{MU}$ has its own structural derivation. The quantity $-\ln \text{MU}/\text{MU}$ is the *Bregman divergence* of μ_1 from μ_0 rescaled by MU^{-1} :

$$-\frac{\ln \text{MU}}{\text{MU}} = -\frac{\ln(\mu_1/\mu_0)}{\mu_1/\mu_0} = \frac{\mu_0}{\mu_1} \ln\left(\frac{\mu_0}{\mu_1}\right). \quad (34)$$

This is the function $f(t) = -t^{-1} \ln t$ evaluated at $t = \text{MU} = \mu_1/\mu_0$. Since $f(t)$ is the derivative of the entropy function at t , there is a mild information-theoretic hook: $-\ln \text{MU}/\text{MU}$ is the “self-information” of the event $\{x = \text{MU}\}$ under the distribution $\rho(x)/\mu_0$, scaled by μ_0/μ_1 . However, this hook remains interpretive rather than geometric.

7.5 Conjecture D: moment-theoretic interpretation

Conjecture D can be stated equivalently as a condition on m_d :

$$\frac{m_d}{m_e} = \exp(\text{MU} E_d - \text{MU} \pi) = \exp(\ln \mu_1 - \text{MU} \pi) = \frac{\mu_1}{\exp(\text{MU} \pi)}. \quad (35)$$

This is the *first moment of ρ , deflated by the exponential of $\text{MU} \pi$* . The down quark mass is therefore:

$$m_d = m_e \cdot \frac{\mu_1}{\exp(\text{MU} \pi)}. \quad (36)$$

Structural interpretation attempt. The quantity $\exp(\text{MU} \pi)$ is the “zero-energy” mass scale in the TOE: $\exp(\text{MU} \cdot \pi) = m_{E=\pi}/m_e$ (the mass at the *edge-sector energy* $E = \pi$, one step below the electron mass in the sector-energy hierarchy). Rewriting:

$$m_d = m_e \cdot \mu_1 \cdot \exp(-\text{MU} \pi) = \frac{\mu_1}{m_{E=\pi}/m_e} \cdot m_e = \frac{\mu_1 \cdot m_e^2}{m_{E=\pi}}. \quad (37)$$

This says: the down quark mass is the first moment of ρ , measured in units of “one electron mass below the electron scale.”

At inverse temperature $\beta = \text{MU}$ (the natural TOE temperature), the Boltzmann weight of the energy $E_d = G_1$ is:

$$e^{-\text{MU}G_1} = e^{-\ln \mu_1} = \frac{1}{\mu_1}. \quad (38)$$

So the Boltzmann weight of the down quark energy at $\beta = \text{MU}$ is the *reciprocal of the first moment of ρ* . Similarly, $e^{-\text{MU}G_0} = 1/\mu_0$: the Boltzmann weight of the top-charm gap at $\beta = \text{MU}$ is the fine-structure constant. Both facts hold by construction (they are algebraic identities), but they reveal that the natural inverse temperature of the TOE mass formula ($\beta = \text{MU} = \mu_1/\mu_0$) *selects* the moment-reciprocals as Boltzmann weights for the two conjectured energies.

Verdict (updated). Conjecture D remains **numerological** (mass error -2.2% , well within QCD scheme uncertainty for light quarks). The formula $E_d = G_1$ is moment-theoretically natural and is coupled to Conjecture B by the exact identity (32). If either formula is given a structural derivation, the other will follow once $-\ln \text{MU}/\text{MU}$ is derived as a $J_3(\mathbb{O})$ invariant.

7.6 Updated open problems

The investigations above sharpen the original open problems OP-B and OP-D of Section 6 to:

OP-B' (*$\mathbb{Z}_3$ -intertwiner norm*). In $J_3(\mathbb{O})$, let $\Phi : V_{\omega^2} \rightarrow V_1$ be the Jordan product $\Phi(y) = \{\alpha_3, y, \alpha_3\}$ (the Jordan triple product restricted to $V_{\omega^2} \ni y, V_1 \ni \alpha_3$). Does $\|\Phi\|_{\text{Tr}}^2 = \mu_0$? If yes, Conjecture B is structural.

OP-D' (*Reduction to OP-B'*). Given OP-B', does $-\ln \text{MU}/\text{MU}$ arise as a natural $J_3(\mathbb{O})$ invariant (e.g., the difference of log-norms of V_{ω^2} and V_ω under the $J_3(\mathbb{O})$ trace form)? If yes, Conjecture D is structural.

OP-B'' (*$m_t/m_c = \alpha^{-1}$ via CKM structure*). An independent route: the top/charm ratio appears in the CKM element V_{cb} . Does the G_2 half-alcove structure (which gives θ_C from Proposition 3.2) also constrain $|V_{cb}|$ in a way that forces $m_t/m_c = \alpha^{-1}$ via the GIM mechanism? This would be a structural derivation via flavor physics rather than $J_3(\mathbb{O})$ geometry.

References

- [1] L. F. Vlegels, *Paper II: The Fine-Structure Constant from S^3 Geometry*, TOE Corpus, 2024.
- [2] L. F. Vlegels, *Paper XVII: Mass Formula from Moment Hierarchy*, TOE Corpus, 2024.
- [3] L. F. Vlegels, *Addendum 36: Sector-Energy Mass Formula — Charged Leptons*, TOE Corpus, 2026.
- [4] L. F. Vlegels, *Addendum 37: Gauge Group from $(B^4, S^3, \mathbb{O})$ Geometry*, TOE Corpus, 2026.
- [5] L. F. Vlegels, *Addendum 38: Quark Sector and Neutrino Structure from $J_3(\mathbb{O})$ Geometry*, TOE Corpus, 2026.
- [6] L. F. Vlegels, *Addendum 39: Higgs Sector Identification from $\mathbb{C} \otimes \mathbb{O}$ Decomposition*, TOE Corpus, 2026.
- [7] L. F. Vlegels, *Addendum 40: $J_3(\mathbb{O})$ Spectral Table, PMNS Corrections, and Higgs VEV Candidate*, TOE Corpus, 2026.

- [8] L. F. Vlegels, *Addendum 41: Strange and Up Quark Energies from the Fano–Octonion Structure*, TOE Corpus, 2026.
- [9] L. F. Vlegels, *Addendum 42: Weinberg Angle from Sector-Fraction Geometry*, TOE Corpus, 2026.
- [10] L. F. Vlegels, *Addendum 43: CFP–Noether Derivation*, TOE Corpus, 2026.
- [11] R. L. Workman et al. (Particle Data Group), *Review of Particle Physics*, Prog. Theor. Exp. Phys. **2022**, 083C01 (2022), updated 2024.