

# $J_3(\mathbb{O})$ Spectral Table, PMNS Corrections, and Higgs VEV Candidate

Addendum 40 to the TOE Corpus

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## Abstract

We complete three tasks that follow directly from the  $J_3(\mathbb{O})$  geometry established in Addenda 37–39. **(i) Spectral table.** The 27-dimensional Albert algebra  $J_3(\mathbb{O})$  decomposes under  $SU(3)_c \times SU(2)_L \times U(1)_Y$  into 27 slots corresponding to three generations of fermions, the gauge bosons, and the Higgs doublet. We tabulate all 27 slots with their SM quantum numbers and the known TOE energy anchors from Addenda 36–39. **(ii) PMNS phase 3.5.** The tribimaximal baseline of Addendum 38 (Proposition 4.1) has three deviations from the PDG mixing angles. We propose three conjectures, all derived from the Cabibbo angle  $\theta_C = \pi / \dim G_2 = \pi/14$  (Conjecture C of Addendum 38) and the  $G_2$  dimension:  $\sin^2 \theta_{13} = \frac{4}{9} \sin^2(\pi/14)$  at +0.26% (**Conjecture F**),  $\sin^2 \theta_{23} = \frac{15}{28}$  at –1.88% (**Conjecture G**), and  $\sin^2 \theta_{12} = \cos^2(\pi/14)/3$  at +3.20% (**Conjecture H**). **(iii) VEV candidate (OP1).** The Higgs vacuum expectation value satisfies  $v = e \cdot M_Z$  to 0.67%, where  $e = \exp(1)$  and  $M_Z = 91.188 \text{ GeV}$ . We record this as a candidate formula pending a structural derivation from the  $J_3(\mathbb{O})$  fold-map spectral problem.

## 1 Introduction and notation

Addendum 37 derived the gauge group  $SU(3)_c \subset G_2 = \text{Aut}(\mathbb{O})$ ,  $SU(2)_L$  as the left isometry of  $S^3$ , and  $U(1)_Y$  as the structure group of the Hopf fiber. Addendum 38 identified quark energies (Conjectures A–E) and the tribimaximal neutrino mixing baseline from the  $\mathbb{Z}_3$  symmetry of the  $J_3(\mathbb{O})$  diagonal. Addendum 39 identified the Higgs doublet as the  $SU(3)_c$ -singlet subspace  $V_H = \mathbb{C}e_0 \oplus \mathbb{C}e_7$  of the Cayley–Dickson decomposition  $\mathbb{C} \otimes \mathbb{O} = \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$ .

Throughout we use the TOE constants

$$\mu_0 = 4\pi^3 + \pi^2 + \pi \approx 137.036, \quad \mu_1 = \frac{16\pi^3}{5} + \frac{3\pi^2}{4} + \frac{2\pi}{3} \approx 108.717, \quad \text{MU} = \frac{\mu_1}{\mu_0} \approx 0.7933, \quad (1)$$

and the mass formula  $m/m_e = \exp(\text{MU}(E - \pi))$  with  $m_e = 0.511 \text{ MeV}$  (Paper 17). The Cabibbo angle from Addendum 38 Conjecture C is

$$\theta_C = \frac{\pi}{\dim G_2} = \frac{\pi}{14}, \quad \sin \theta_C = \sin \frac{\pi}{14} \approx 0.22252. \quad (2)$$

## 2 The $J_3(\mathbb{O})$ spectral table

### 2.1 Decomposition structure

An element of  $J_3(\mathbb{O})$  has the form

$$X = \begin{pmatrix} \alpha_1 & x_{12} & x_{13} \\ \bar{x}_{12} & \alpha_2 & x_{23} \\ \bar{x}_{13} & \bar{x}_{23} & \alpha_3 \end{pmatrix}, \quad \alpha_i \in \mathbb{R}, \quad x_{ij} \in \mathbb{O}. \quad (3)$$

The three diagonal entries  $\alpha_1, \alpha_2, \alpha_3$  are  $\mathbb{Z}_3$ -symmetric generation labels (Addendum 38, Section 3). Each off-diagonal entry  $x_{ij} \in \mathbb{O}$  has 8 real components, labelled by the octonion basis  $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$  where  $e_0 = 1$  is the real unit and  $e_7$  is the  $G_2$ -stabilized imaginary unit.

Under the decomposition  $\mathbb{C} \otimes \mathbb{O} \cong \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$  (Addendum 39, Theorem 2.1):

- $e_0$ :  $SU(3)_c$  singlet,  $SU(2)_L$  singlet,  $Y = -1$  (Higgs or gauge boson slot).
- $e_7$ :  $SU(3)_c$  singlet,  $SU(2)_L$  singlet,  $Y = -1/3$  (CKM/Cabibbo mixing direction).
- $e_1, e_2, e_3$ : colour triplet  $\mathbf{3}$ ,  $SU(2)_L$  doublet component,  $Y = +1/6$  (up-type quarks).
- $e_4, e_5, e_6$ : colour anti-triplet  $\bar{\mathbf{3}}$ ,  $SU(2)_L$  singlet,  $Y = -1/3$  (down-type quarks).

## 2.2 The 27-slot table

Table 1 lists all 27 degrees of freedom of  $J_3(\mathbb{O})$ , together with the SM particle or role identified with each slot, the known TOE energy anchor, and the predicted mass against the PDG value.

Table 1:  $J_3(\mathbb{O})$  spectral table.  $E$  = TOE energy parameter,  $m_{\text{calc}}$  from  $m = m_e \exp(\text{MU}(E - \pi))$ . Slots with “—” have no clean TOE formula yet.

| #                                                                       | Slot               | SU(3) <sub>c</sub> | SM particle            | $E$ -formula                         | $m_{\text{calc}}$ | PDG       | err      |
|-------------------------------------------------------------------------|--------------------|--------------------|------------------------|--------------------------------------|-------------------|-----------|----------|
| <i>Diagonal — generation labels</i>                                     |                    |                    |                        |                                      |                   |           |          |
| 1                                                                       | $\alpha_1$         | singlet            | electron               | $\pi$                                | 0.511 MeV         | 0.511 MeV | exact    |
| 2                                                                       | $\alpha_2$         | singlet            | muon                   | $\pi^2$                              | 106.3 MeV         | 105.7 MeV | +0.60%   |
| 3                                                                       | $\alpha_3$         | singlet            | tau                    | $E_\tau$                             | 1777 MeV          | 1777 MeV  | +0.08%   |
| <i>Off-diagonal block <math>x_{12}</math> — first/second generation</i> |                    |                    |                        |                                      |                   |           |          |
| 4                                                                       | $x_{12} \cdot e_0$ | singlet            | $W / \nu_e$ doublet    | —                                    | —                 | 80.4 GeV  | —        |
| 5                                                                       | $x_{12} \cdot e_7$ | singlet            | $d$ - $s$ Cabibbo dir. | $\ln(\mu_1)/\text{MU}$               | 4.60 MeV          | 4.67 MeV  | −1.6%    |
| 6                                                                       | $x_{12} \cdot e_1$ | <b>3</b>           | $u_r$                  | —                                    | —                 | 2.16 MeV  | —        |
| 7                                                                       | $x_{12} \cdot e_2$ | <b>3</b>           | $u_g$                  | —                                    | —                 | 2.16 MeV  | —        |
| 8                                                                       | $x_{12} \cdot e_3$ | <b>3</b>           | $u_b$                  | —                                    | —                 | 2.16 MeV  | —        |
| 9                                                                       | $x_{12} \cdot e_4$ | $\bar{\mathbf{3}}$ | $d_r$                  | $\ln(\mu_1)/\text{MU}$               | 4.60 MeV          | 4.67 MeV  | −1.6%    |
| 10                                                                      | $x_{12} \cdot e_5$ | $\bar{\mathbf{3}}$ | $d_g$                  | $\ln(\mu_1)/\text{MU}$               | 4.60 MeV          | 4.67 MeV  | −1.6%    |
| 11                                                                      | $x_{12} \cdot e_6$ | $\bar{\mathbf{3}}$ | $d_b$                  | $\ln(\mu_1)/\text{MU}$               | 4.60 MeV          | 4.67 MeV  | −1.6%    |
| <i>Off-diagonal block <math>x_{13}</math> — first/third generation</i>  |                    |                    |                        |                                      |                   |           |          |
| 12                                                                      | $x_{13} \cdot e_0$ | singlet            | $Z / \nu_\tau$ mix     | —                                    | —                 | 91.2 GeV  | —        |
| 13                                                                      | $x_{13} \cdot e_7$ | singlet            | $d$ - $b$ CKM dir.     | $\ln(\mu_1)/\text{MU}$               | 4.60 MeV          | 4.67 MeV  | −1.6%    |
| 14                                                                      | $x_{13} \cdot e_1$ | <b>3</b>           | $c_r$                  | $\pi^2 + \pi$                        | 1285 MeV          | 1270 MeV  | +1.2%    |
| 15                                                                      | $x_{13} \cdot e_2$ | <b>3</b>           | $c_g$                  | $\pi^2 + \pi$                        | 1285 MeV          | 1270 MeV  | +1.2%    |
| 16                                                                      | $x_{13} \cdot e_3$ | <b>3</b>           | $c_b$                  | $\pi^2 + \pi$                        | 1285 MeV          | 1270 MeV  | +1.2%    |
| 17                                                                      | $x_{13} \cdot e_4$ | $\bar{\mathbf{3}}$ | $s_r$                  | —                                    | —                 | 93.4 MeV  | —        |
| 18                                                                      | $x_{13} \cdot e_5$ | $\bar{\mathbf{3}}$ | $s_g$                  | —                                    | —                 | 93.4 MeV  | —        |
| 19                                                                      | $x_{13} \cdot e_6$ | $\bar{\mathbf{3}}$ | $s_b$                  | —                                    | —                 | 93.4 MeV  | —        |
| <i>Off-diagonal block <math>x_{23}</math> — second/third generation</i> |                    |                    |                        |                                      |                   |           |          |
| 20                                                                      | $x_{23} \cdot e_0$ | singlet            | Higgs / VEV            | OP1 cand.                            | 246 GeV           | 246 GeV   | (−0.67%) |
| 21                                                                      | $x_{23} \cdot e_7$ | singlet            | $s$ - $b$ CKM dir.     | —                                    | —                 | —         | —        |
| 22                                                                      | $x_{23} \cdot e_1$ | <b>3</b>           | $t_r$                  | $\pi^2 + \pi + \ln(\mu_0)/\text{MU}$ | 176 GeV           | 173 GeV   | +1.9%    |
| 23                                                                      | $x_{23} \cdot e_2$ | <b>3</b>           | $t_g$                  | $\pi^2 + \pi + \ln(\mu_0)/\text{MU}$ | 176 GeV           | 173 GeV   | +1.9%    |
| 24                                                                      | $x_{23} \cdot e_3$ | <b>3</b>           | $t_b$                  | $\pi^2 + \pi + \ln(\mu_0)/\text{MU}$ | 176 GeV           | 173 GeV   | +1.9%    |
| 25                                                                      | $x_{23} \cdot e_4$ | $\bar{\mathbf{3}}$ | $b_r$                  | $\pi^{7/3}$                          | 4040 MeV          | 4180 MeV  | −3.3%    |
| 26                                                                      | $x_{23} \cdot e_5$ | $\bar{\mathbf{3}}$ | $b_g$                  | $\pi^{7/3}$                          | 4040 MeV          | 4180 MeV  | −3.3%    |
| 27                                                                      | $x_{23} \cdot e_6$ | $\bar{\mathbf{3}}$ | $b_b$                  | $\pi^{7/3}$                          | 4040 MeV          | 4180 MeV  | −3.3%    |

**Remark 2.1.** The 27 slots account for all SM matter content: three charged leptons (diagonal), nine up-type quarks (**3** entries in each block), nine down-type quarks ( $\bar{\mathbf{3}}$  entries), three CKM/Cabibbo directions ( $e_7$  entries), and three electroweak gauge boson / Higgs slots ( $e_0$  entries). Neutrinos do not appear as independent  $J_3(\mathbb{O})$  slots in this decomposition; they arise as the SU(2)<sub>L</sub> doublet partners of the charged leptons, consistent with the Type 1 mass gap of Addendum 38 Proposition 3.2.

## 2.3 Derivation status of occupied slots

Of the 27 slots, 16 carry known TOE energy formulas (slots 1–3, 5, 9–11, 13–16, 22–27), 4 carry candidate formulas (slot 2 at +0.60%, slot 20 as OP1, slots 25–27 from Fano conjecture), and 7 remain open (slots 4, 6–8, 12, 17–19, 21). The open slots correspond to the  $u$  quark,  $s$  quark,  $W$  and  $Z$  bosons, and two CKM off-diagonal directions.

### 3 Phase 3.5: PMNS mixing angles from Cabibbo and $G_2$

#### 3.1 TBM baseline and its deviations

Addendum 38 Proposition 4.1 derived the tribimaximal (TBM) mixing baseline

$$\sin^2 \theta_{12}^{\text{TBM}} = \frac{1}{3}, \quad \sin^2 \theta_{23}^{\text{TBM}} = \frac{1}{2}, \quad \theta_{13}^{\text{TBM}} = 0 \quad (4)$$

from the  $\mathbb{Z}_3$  symmetry of the  $J_3(\mathbb{O})$  diagonal via  $\pi_1(L(3, 1)) \cong \mathbb{Z}_3$ . Against the PDG 2024 values ( $\sin^2 \theta_{12} = 0.307$ ,  $\sin^2 \theta_{23} = 0.546$ ,  $\sin^2 \theta_{13} = 0.02195$ ), the TBM deviations are +8.6%, −8.4%, and −100% respectively.

The TBM corrections arise from the off-diagonal  $J_3(\mathbb{O})$  phases under  $G_2$  action. Two inputs are available from the corpus: the Cabibbo angle  $\theta_C = \pi/14$  (Addendum 38 Conj. C, −0.80% from  $|V_{us}|$ ) and the  $G_2$  dimension  $\dim G_2 = 14$ .

#### 3.2 Reactor angle (Conjecture F)

**Conjecture 3.1** (PMNS reactor angle).

$$\sin^2 \theta_{13} = \frac{4}{9} \sin^2 \frac{\pi}{14} = \frac{4}{9} \sin^2 \theta_C. \quad (5)$$

*Numerical check.*  $\frac{4}{9} \sin^2(\pi/14) = \frac{4}{9} \times 0.049516 = 0.022007$ . PDG: 0.02195. Error: +0.26%.

*Structural interpretation.* The factor  $2/3$  in  $\sin \theta_{13} = \frac{2}{3} \sin \theta_C$  equals the hypercharge of the up-type quarks,  $Y_u = +2/3$ . In the  $J_3(\mathbb{O})$  off-diagonal blocks, the Cabibbo rotation lives in the  $e_7$  direction (quark sector); the reactor angle  $\theta_{13}$  couples the lepton singlet ( $e_0$  slot) to the quark triplet. The hypercharge suppression factor  $Y_u = 2/3$  converts the quark-sector Cabibbo rotation to the lepton-sector reactor angle, realising a form of quark–lepton unification through the  $J_3(\mathbb{O})$  structure.

Equivalently,  $\sin^2 \theta_{13} / \sin^2 \theta_C = 4/9 = Y_u^2$ , so the reactor angle is the Cabibbo angle attenuated by the square of the up-quark hypercharge.

#### 3.3 Atmospheric mixing (Conjecture G)

**Conjecture 3.2** (PMNS atmospheric angle).

$$\sin^2 \theta_{23} = \frac{15}{28} = \frac{1}{2} + \frac{1}{2 \dim G_2}. \quad (6)$$

*Numerical check.*  $15/28 \approx 0.53571$ . PDG: 0.546. Error: −1.88%.

*Structural interpretation.* The TBM baseline  $\sin^2 \theta_{23} = 1/2$  is maximal mixing. The  $G_2$  correction  $+1/(2 \dim G_2) = +1/28$  is the first-order symmetry breaking from  $G_2 \supset \text{SU}(3)_c$  (Addendum 37, Theorem 4.1). Geometrically,  $G_2$  has 14 generators; the  $\text{SU}(3)$  subalgebra uses 8, leaving  $14 - 8 = 6$  generators that mix the  $\mathbf{3}$  and  $\bar{\mathbf{3}}$  directions. The correction  $1/(2 \times 14) = 1/28$  reflects the density of these cross-sector generators relative to the full  $G_2$  algebra.

**Remark 3.3.** The sign of the correction (+, lifting  $\theta_{23}$  above maximal) is consistent with the PDG central value, which exceeds  $45^\circ$ . The error of −1.88% is larger than Conjecture F but still within two standard deviations of the current experimental uncertainty on  $\sin^2 \theta_{23}$ .

#### 3.4 Solar mixing (Conjecture H)

**Conjecture 3.4** (PMNS solar angle).

$$\sin^2 \theta_{12} = \frac{\cos^2(\pi/14)}{3} = \frac{1 - \sin^2 \theta_C}{3}. \quad (7)$$

*Numerical check.*  $\cos^2(\pi/14)/3 = (1 - 0.049516)/3 = 0.31683$ . PDG: 0.307. Error: +3.20%.

*Structural interpretation.* The TBM solar angle  $1/3$  is the uniform distribution of the first-generation lepton amplitude across three neutrino mass eigenstates ( $\mathbb{Z}_3$  symmetry). The Cabibbo correction reduces this to  $\cos^2(\theta_C)/3$ : the Cabibbo rotation in the quark sector “borrows” a fraction  $\sin^2 \theta_C$  of the mixing amplitude, leaving  $(1 - \sin^2 \theta_C)/3$  in the lepton sector. This is a specific realisation of quark–lepton complementarity ( $\theta_{12} + \theta_C \approx \pi/4$ ) within the  $J_3(\mathbb{O})$  framework.

**Remark 3.5.** The 3.20% error is larger than Conjectures F and G. The  $\theta_{12}$  prediction should be treated as a weaker conjecture pending a direct derivation from the  $G_2$  action on  $J_3(\mathbb{O})$  off-diagonal phases. Nevertheless, the formula  $\sin^2 \theta_{12} = (1 - \sin^2 \theta_C)/3$  has no free parameters: it uses only the Cabibbo angle from Addendum 38 Conjecture C and the TBM  $1/3$  factor.

### 3.5 Summary of Phase 3.5 conjectures

Table 2: PMNS mixing angle conjectures (Phase 3.5). All formulas use  $\theta_C = \pi/14$  from Addendum 38 Conj. C and  $\dim G_2 = 14$  from Addendum 37 Theorem 4.1.

| Label   | Angle         | Formula                | Predicted $\sin^2$ | PDG     | Error  | Status   |
|---------|---------------|------------------------|--------------------|---------|--------|----------|
| Conj. F | $\theta_{13}$ | $(4/9) \sin^2(\pi/14)$ | 0.02201            | 0.02195 | +0.26% | strong   |
| Conj. G | $\theta_{23}$ | $15/28$                | 0.53571            | 0.546   | −1.88% | moderate |
| Conj. H | $\theta_{12}$ | $\cos^2(\pi/14)/3$     | 0.31683            | 0.307   | +3.20% | weak     |
| TBM     | $\theta_{12}$ | $1/3$                  | 0.33333            | 0.307   | +8.6%  | baseline |
| TBM     | $\theta_{23}$ | $1/2$                  | 0.50000            | 0.546   | −8.4%  | baseline |
| TBM     | $\theta_{13}$ | 0                      | 0                  | 0.02195 | −100%  | baseline |

All three conjectures improve significantly over the TBM baseline without introducing any free parameters. Conjecture F achieves sub-percent agreement and is the highest-confidence result of this addendum.

## 4 OP1: Higgs VEV candidate

Addendum 39 identified the Higgs doublet slot as  $V_H = \mathbb{C}e_0 \oplus \mathbb{C}e_7$  (slot 20 in Table 1) but left the vacuum expectation value  $v$  as Open Problem 1 (OP1). The slot 20 energy is

$$E(v) = \pi + \frac{\ln(v/m_e)}{\text{MU}} \approx 19.636. \quad (8)$$

**Conjecture 4.1** (Higgs VEV, OP1 candidate).

$$v = e \cdot M_Z, \quad (9)$$

where  $e = \exp(1) \approx 2.7183$  and  $M_Z = 91.188 \text{ GeV}$ .

*Numerical check.*  $e \times 91.188 = 247.87 \text{ GeV}$ . PDG:  $v = 246.22 \text{ GeV}$ . Error: −0.67%.

*Structural interpretation.* In the TOE,  $e = \exp(1)$  is the natural scale factor for first moments (Paper 17: the first-moment coupling is  $\mu_1$ , computed from  $\rho(x)$  via  $\mu_1 = \int_0^1 x\rho(x) dx$ ). The  $Z$  boson sits in the  $e_0$  slot of block  $x_{13}$  (slot 12), the same  $e_0$  direction as the Higgs/VEV slot (slot 20) but in the adjacent off-diagonal block. A shift by one  $J_3(\mathbb{O})$  block in the  $e_0$  direction by the Euler scale factor  $e$  would yield  $E_v = E_Z + 1/\text{MU} = 18.384 + 1.260 = 19.644$ , which is close to but not exactly  $E(v) = 19.636$ .

The formula  $v = e \cdot M_Z$  is therefore recorded as a *numerical candidate*, not a structural derivation. A proof would require identifying the mechanism by which the  $e_0$  Higgs slot acquires a VEV scaled by  $\exp(1)$  relative to the  $Z$  mass — the fold-map spectral problem referenced in Addendum 39 OP1.

**Remark 4.2.** The three candidates investigated for OP1 in the search (Addendum 39, Open Problem 1) were: (a)  $v = e \cdot M_Z$  at  $-0.67\%$ ; (b)  $v$  from a  $J_3(\mathbb{O})$  spectral eigenvalue (not yet identified); and (c)  $v = M_Z / \cos \theta_W$  at tree level, which with  $\sin^2 \theta_W = 1/(1+\pi)$  (Addendum 37 Conjecture 5.3) gives  $v_{\text{tree}} = M_Z / \sqrt{1 - 1/(1+\pi)} = M_Z \sqrt{(1+\pi)/\pi} = 91.188 \times 2.032 = 185.3 \text{ GeV}$  — well outside the observed value, ruling this route out at tree level.

## 5 Open problems

The spectral table and Phase 3.5 work leaves the following problems open.

**OP1 (VEV).** Derive  $v$  from the  $J_3(\mathbb{O})$  fold-map spectral problem rather than from the numerical coincidence  $v \approx e \cdot M_Z$ .

**OP2 ( $u$  quark).** No clean TOE formula found for the  $u$ -quark energy  $E_u \approx 4.959$ . The formula  $E_u = \pi^2/2 = 4.935$  gives  $-0.49\%$  but lacks a structural derivation. The  $u$ -quark slot (6–8) is the up-type quark in the  $x_{12}$  block; its energy should be expressible in terms of  $\pi$ ,  $\mu_0$ ,  $\mu_1$  without free parameters.

**OP3 ( $s$  quark).** The  $s$ -quark energy  $E_s \approx 9.707$  has no clean formula. The strange quark sits in slots 17–19 ( $\bar{\mathbf{3}}$  in block  $x_{13}$ ), the down-type companion of the charm quark. A formula relating  $E_s$  to  $E_d$  and  $E_b$  via the Fano ratio  $\pi^{N_{\text{Im}}/N_c}$  is a natural target.

**OP4 ( $W$  boson).** The  $W$  mass  $M_W \approx 80.4 \text{ GeV}$  sits in the  $e_0$  slot of block  $x_{12}$  (slot 4). Its relationship to  $M_Z$  via the Weinberg angle  $\sin^2 \theta_W = 1/(1+\pi)$  (Addendum 37 Conjecture 5.3) is  $M_W = M_Z \cos \theta_W = 91.188 \times 0.871 = 79.4 \text{ GeV}$  at  $-1.2\%$ , matching to within  $1\%$ . However, Conjecture 5.3 requires RG running verification.

**OP5 (exact PMNS angles).** Conjectures G and H have errors of  $-1.88\%$  and  $+3.20\%$ . The exact  $\theta_{23}$  and  $\theta_{12}$  values require a complete treatment of the  $G_2$  action on  $J_3(\mathbb{O})$  off-diagonal phases (Phase 3.5 completion, pending Phase 4 completion of the full  $J_3(\mathbb{O})$  spectrum).

**OP6 (muon 0.60% residual).** As noted in Addendum 38 Proposition 2.1, the muon energy  $E_\mu = \pi^2$  (cruciform identification on  $C_2$ ) overshoots by  $0.60\%$  because MU uses bulk-inclusive  $\mu_1$ , while the muon propagates on the boundary-only  $C_2 \subset S^3$ . Deriving  $\text{MU}^{(S^3)}$  for propagation restricted to  $S^3$  would close this gap.

## 6 Summary

This addendum completes three intermediate goals of the TOE SM-completion programme:

1. **Spectral table** (Section 2): the full 27-slot  $J_3(\mathbb{O})$  decomposition is tabulated with SM quantum numbers. Of 27 slots, 16 carry derived TOE energy anchors. Seven slots ( $u$  quark,  $s$  quark,  $W/Z$  bosons, two CKM directions) remain open.
2. **PMNS Phase 3.5** (Section 3): three conjectures derived from the Cabibbo angle  $\theta_C = \pi/14$  and  $\dim G_2 = 14$ : Conjecture F ( $\theta_{13}$ ,  $+0.26\%$ , strong), Conjecture G ( $\theta_{23}$ ,  $-1.88\%$ , moderate), and Conjecture H ( $\theta_{12}$ ,  $+3.20\%$ , weak). All improve on the TBM baseline.

3. **VEV candidate** (Section 4):  $v = e \cdot M_Z$  at  $-0.67\%$ , recorded as a numerical candidate for OP1 pending structural derivation.

The dominant open problem remains the derivation of the Higgs VEV and the  $W/Z$  masses from first principles in the  $J_3(\mathbb{O})$  fold-map spectral problem. Once the gauge boson masses are derived, the full SM spectrum will be accounted for by the TOE corpus.

## References

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