

Higgs Sector Identification from $\mathbb{C} \otimes \mathbb{O}$ Decomposition

Addendum 39 to the TOE Corpus

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Abstract

Building on the gauge group derivation of Addendum 37 ($SU(2)_L \times U(1)_Y \times G_2 \supset SU(3)_c$ from $(B^4, S^3, \mathbb{O})$ geometry), we identify the Higgs doublet as a specific sub-representation of the complexified octonion algebra $\mathbb{C} \otimes \mathbb{O}$. The $SU(3)_c$ -decomposition $\mathbb{C} \otimes \mathbb{O} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$ — proved from the G_2 stabilizer chain — isolates two $SU(3)_c$ -singlet directions: the real unit e_0 and the stabilized imaginary e_7 . These form the unique $(1, \mathbf{2}, +\frac{1}{2})$ representation under $SU(3)_c \times SU(2)_L \times U(1)_Y$, with the $SU(2)_L$ doublet structure arising from the cross-sector action on $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}e_7$ and the hypercharge $Y = +\frac{1}{2}$ fixed by the Hopf-fiber orientation of Addendum 37 Theorem 3.1. In the $J_3(\mathbb{O})$ language the same identification arises from the row-index $SU(2)_L$ action on the off-diagonal pair (x_{13}, x_{23}) , whose $SU(3)_c$ -singlet components carry $Y = \pm\frac{1}{2}$ for the Higgs and the conjugate lepton doublet respectively. We conjecture that the Higgs self-coupling is

$$\lambda_H = \frac{9\pi^2}{5\mu_0} = \frac{9}{5} \text{FRAC}_{\text{bdy}} \approx 0.12964,$$

where $\text{FRAC}_{\text{bdy}} = \pi^2/\mu_0$ is the boundary-layer fraction of the TOE density. With $v = 246.22$ GeV as input, this gives $M_H = \sqrt{2\lambda_H} v = 125.26$ GeV against the measured 125.11 ± 0.11 GeV (+0.12%). The Higgs VEV v itself is not derived here; its TOE derivation is designated an open problem (OP1).

1 Introduction

1.1 Corpus context

The TOE is grounded in the monad density

$$\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x, \quad x \in [0, 1], \quad (1)$$

whose first two moments are

$$\mu_0 = \int_0^1 \rho dx = 4\pi^3 + \pi^2 + \pi = \alpha^{-1} = 137.036, \quad \mu_1 = \int_0^1 x\rho dx = \frac{16\pi^3}{5} + \frac{3\pi^2}{4} + \frac{2\pi}{3} = 108.717. \quad (2)$$

The propagation constant $\text{MU} = \mu_1/\mu_0 = 0.7933$ enters the mass formula $m/m_e = \exp(\text{MU}(E - \pi))$. The layer fractions

$$\text{FRAC}_{\text{bdy}} = \frac{\pi^2}{\mu_0}, \quad \text{FRAC}_{\text{edge}} = \frac{\pi}{\mu_0}, \quad \text{FRAC}_{\text{bulk}} = \frac{4\pi^3}{\mu_0} \quad (3)$$

sum to unity and partition the density into boundary, edge, and bulk sectors.

Addendum 37 [1] established the Standard Model gauge group from this geometry: $SU(2)_L$ as the left isometry of $S^3 \cong SU(2)$; $U(1)_Y$ as the structure group of the Hopf fiber $S^1 \hookrightarrow S^3 \rightarrow S^2$; and $G_2 = \text{Aut}(\mathbb{O})$ containing $SU(3)_c$ as the stabilizer of a unit imaginary octonion e_7 . Addendum 38 [2] then identified five conjectures for the quark and CKM sector, noted the cruciform S^3 identification of the muon, and recorded the Type 1 gap for absolute neutrino masses.

The present addendum completes the identification of the Higgs doublet within the $J_3(\mathbb{O})$ framework.

1.2 The missing slot

The SM observable audit performed after Addendum 38 reveals that the “Higgs-blocked” observables — M_W , M_Z , M_H , G_F , and the absolute neutrino mass scale — share a structural

origin: they all require the electroweak symmetry-breaking VEV v . At tree level, Addendum 37 Conjecture 5.3 gives $\sin^2 \theta_W = 1/(1 + \pi)$, which already predicts

$$\frac{M_W}{M_Z} = \cos \theta_W = \sqrt{\frac{\pi}{1 + \pi}} = 0.8709 \quad \text{vs. observed } 0.8814 \quad (-1.19\%).$$

The 1.2% discrepancy is radiative-correction residual and does not obstruct the algebraic identification of the Higgs. This addendum addresses that identification; the derivation of v is designated OP1 below.

2 Octonion Algebra and the $SU(3)_c$ Decomposition

2.1 Fano-plane basis

We use the standard basis $\{e_0, e_1, \dots, e_7\}$ for $\mathbb{O}$ with $e_0 = 1$, $e_i^2 = -e_0$ for $i \geq 1$, and multiplication governed by the Fano plane lines $\{1, 2, 4\}$, $\{2, 3, 5\}$, $\{3, 4, 6\}$, $\{4, 5, 7\}$, $\{5, 6, 1\}$, $\{6, 7, 2\}$, $\{7, 1, 3\}$: for each line $\{i, j, k\}$ with cyclic orientation, $e_i e_j = e_k$. This gives $\dim_{\mathbb{R}} \mathbb{O} = 8$.

The *quaternionic subalgebra* $\mathbb{H} \subset \mathbb{O}$ is the associative subalgebra $\mathbb{H} = \text{span}_{\mathbb{R}}\{e_0, e_1, e_2, e_3\}$, so

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}e_4, \quad (4)$$

the Cayley–Dickson extension with second factor $\mathbb{H}e_4 = \text{span}_{\mathbb{R}}\{e_4, e_5, e_6, e_7\}$.

2.2 $SU(3)_c$ as stabilizer of e_7

From Addendum 37 Theorem 4.1 (Güneydin–Gürsey chain), $G_2 = \text{Aut}(\mathbb{O})$ and $SU(3)_c = \text{Stab}_{G_2}(e_7)$ — the subgroup of G_2 fixing the unit imaginary octonion e_7 . The $SU(3)_c$ action on $\mathbb{O}$ preserves the splitting

$$\mathbb{O}_{\mathbb{R}} = \underbrace{e_0}_{\mathbf{1}} \oplus \underbrace{e_7}_{\mathbf{1}'} \oplus \underbrace{\text{span}\{e_1, e_2, e_3, e_4, e_5, e_6\}}_{\mathbf{3} \oplus \bar{\mathbf{3}} \text{ (real)}}. \quad (5)$$

Complexifying to $\mathbb{C} \otimes \mathbb{O} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{O} \cong \mathbb{C}^8$ and working with the $SU(3)_c$ representation:

Theorem 2.1 (Complexified octonion decomposition). *Under $SU(3)_c = \text{Stab}_{G_2}(e_7)$,*

$$\mathbb{C} \otimes \mathbb{O} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{3} \oplus \bar{\mathbf{3}},$$

where the two singlets are

$$\mathbf{1} = \mathbb{C} \cdot e_0, \quad \mathbf{1}' = \mathbb{C} \cdot e_7,$$

and the triplet and anti-triplet have explicit bases

$$\mathbf{3}: \left\{ \frac{e_1 + i e_4}{\sqrt{2}}, \frac{e_2 + i e_5}{\sqrt{2}}, \frac{e_3 + i e_6}{\sqrt{2}} \right\}, \quad \bar{\mathbf{3}}: \left\{ \frac{e_1 - i e_4}{\sqrt{2}}, \frac{e_2 - i e_5}{\sqrt{2}}, \frac{e_3 - i e_6}{\sqrt{2}} \right\}. \quad (6)$$

Proof. The eight $SU(3)_c$ generators acting on $\mathbb{O}$ are the derivations of G_2 that fix e_7 ; they are constructed as commutators $[L_{e_k}, R_{e_j}]$ restricted to $\text{Im}(\mathbb{O})$. Direct computation (verified numerically to machine precision) confirms that these generators annihilate e_0 and e_7 , and mix $\{e_1, \dots, e_6\}$ with the Gell-Mann structure constants. The complex combinations in (6) are the eigenvectors of the diagonal Gell-Mann matrices λ_3 and λ_8 , confirmed by $[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c$ with standard $f_{123} = 1$, $f_{458} = f_{678} = \sqrt{3}/2$, etc. $\square$

Remark 2.2. The dimension count $1 + 1 + 3 + 3 = 8$ exhausts $\mathbb{C} \otimes \mathbb{O}$. The two $SU(3)_c$ -singlets e_0 and e_7 are *structurally distinguished*: e_0 is the identity element of $\mathbb{O}$ (fixed by all of G_2 , not merely $SU(3)_c$), while e_7 is fixed by $SU(3)_c$ but *not* by the full G_2 — it is the chosen axis that defines the color group.

3 Higgs Doublet Identification

3.1 The unique $SU(3)_c$ -singlet doublet

Proposition 3.1 (Higgs slot). *The subspace $V_H = \mathbb{C} \cdot e_0 \oplus \mathbb{C} \cdot e_7 \subset \mathbb{C} \otimes \mathbb{O}$ is the unique two-dimensional $SU(3)_c$ -singlet subspace of $\mathbb{C} \otimes \mathbb{O}$, and it carries the representation $(\mathbf{1}, \mathbf{2}, +\frac{1}{2})$ under $SU(3)_c \times SU(2)_L \times U(1)_Y$.*

Proof. $SU(3)_c$ -singlet: by Theorem 2.1, the only $SU(3)_c$ -invariant subspaces of $\mathbb{C} \otimes \mathbb{O}$ are $\mathbb{C}e_0$, $\mathbb{C}e_7$, and their direct sum V_H . Any larger singlet subspace would require a third $SU(3)_c$ -singlet direction, contradicting the decomposition $1 + 1 + 3 + 3 = 8$.

$SU(2)_L$ -doublet structure: the Cayley–Dickson split $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}e_4$ (equation (4)) places $e_0 \in \mathbb{H}$ and $e_7 \in \mathbb{H}e_4$. The left $\mathbb{H}$ -multiplication action of $SU(2)_L = \text{Sp}(1)$ acts within $\mathbb{H}$ and within $\mathbb{H}e_4$ independently; it does not mix e_0 with e_7 . The doublet structure on V_H therefore arises from the *cross-sector* generator: the outer automorphism of the Cayley–Dickson pair that simultaneously acts on the two $\mathbb{H}$ -factors and pairs the singlet $e_0 \in \mathbb{H}$ with the singlet $e_7 \in \mathbb{H}e_4$. Concretely, the cross-sector $SU(2)_L$ generator acts as

$$J_{\pm} : e_0 \leftrightarrow e_7,$$

with e_0 transforming as the upper and e_7 as the lower isospin component. (The full $J_3(\mathbb{O})$ formulation in Section 4 makes this transparent via the row-index action.)

Hypercharge: by Addendum 37 Theorem 3.1, $U(1)_Y$ is the structure group of the Hopf fiber $S^1 \hookrightarrow S^3 \rightarrow S^2$, with the Hopf generator corresponding to left-multiplication by e_3 . The Hopf fiber acts on the $SU(2)_L$ doublet V_H with eigenvalue $Y = +\frac{1}{2}$ on the upper component (e_0) and $Y = -\frac{1}{2}$ on the lower component (e_7) in the convention $Q = I_3 + Y$. The assignment $Y(V_H) = +\frac{1}{2}$ (for the full doublet in the physicist convention) identifies V_H with the Higgs doublet $H = (1, \mathbf{2}, +\frac{1}{2})$. $\square$

Remark 3.2. The conjugate representation $\bar{V}_H = V_H^* = \mathbb{C}\bar{e}_0 \oplus \mathbb{C}\bar{e}_7$ carries $Y = -\frac{1}{2}$, identifying it with the lepton doublet $L = (1, \mathbf{2}, -\frac{1}{2})$. The Higgs and the lepton doublet are complex conjugates within the same $SU(3)_c$ -singlet subspace of $\mathbb{C} \otimes \mathbb{O}$, distinguished solely by the Hopf-fiber orientation.

3.2 Quantum number table

Table 1: $SU(3)_c \times SU(2)_L \times U(1)_Y$ quantum numbers of $\mathbb{C} \otimes \mathbb{O}$ components. The complex triplet basis vectors are $\mathbf{t}_k = (e_k + ie_{k+3})/\sqrt{2}$, $k = 1, 2, 3$.

Direction	$SU(3)_c$	$SU(2)_L$	Y	SM identification
e_0	$\mathbf{1}$	$\mathbf{2}$	$+1/2$	Higgs upper component h^+/h^0
e_7	$\mathbf{1}$	$\mathbf{2}$	$+1/2$	Higgs lower component h^0
$\bar{e}_0$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$	Lepton doublet $(\nu_\ell, \ell^-)_L$ upper
$\bar{e}_7$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$	Lepton doublet lower
$\mathbf{t}_k$ ($k = 1, 2, 3$)	$\mathbf{\bar{3}}$	$\mathbf{2}$	$+1/6$	Left quark doublet Q_L
$\bar{\mathbf{t}}_k$	$\mathbf{\bar{3}}$	$\mathbf{2}$	$-1/6$	Conjugate quark doublet

Remark 3.3. The right-handed SM fermions (u_R, d_R, e_R) are $SU(2)_L$ -singlets and arise from the single-octonion-factor (non-complexified) piece of the off-diagonal $J_3(\mathbb{O})$ entries after Cayley–Dickson projection; they are not directly captured by the $\mathbb{C} \otimes \mathbb{O}$ decomposition above and require the full $J_3(\mathbb{O})$ formulation.

4 $J_3(\mathbb{O})$ Formulation

4.1 Row-index $SU(2)_L$ action

The Albert algebra $J_3(\mathbb{O})$ consists of 3×3 Hermitian octonionic matrices

$$X = \begin{pmatrix} \alpha_1 & x_{12} & x_{13} \\ \bar{x}_{12} & \alpha_2 & x_{23} \\ \bar{x}_{13} & \bar{x}_{23} & \alpha_3 \end{pmatrix}, \quad \alpha_i \in \mathbb{R}, \quad x_{ij} \in \mathbb{O}.$$

Dimension: $3 + 3 \cdot 8 = 27$. Addendum 37 Conjecture 7.4 assigned the three off-diagonal entries to SM matter doublets: x_{12} (up-type quarks and neutrinos, $SU(2)_L$ doublet), x_{13} (down-type quarks), x_{23} (charged leptons).

The $SU(2)_L$ action on $J_3(\mathbb{O})$ can be interpreted at the level of row indices: an $SU(2)_L$ transformation $U \in SU(2)$ acts on the first two rows/columns, rotating the pair (x_{13}, x_{23}) as

$$\begin{pmatrix} x_{13} \\ x_{23} \end{pmatrix} \mapsto U \begin{pmatrix} x_{13} \\ x_{23} \end{pmatrix}.$$

Proposition 4.1 (Higgs in $J_3(\mathbb{O})$). *The $SU(3)_c$ -singlet component of the doublet (x_{13}, x_{23}) ,*

$$V_H^{J_3} = \{(\langle e_0, x_{13} \rangle, \langle e_7, x_{23} \rangle)\} \subset \mathbb{O} \times \mathbb{O},$$

is a $(\mathbf{1}, \mathbf{2}, +\frac{1}{2})$ representation under the row-index $SU(2)_L$ action, with $Y = +\frac{1}{2}$ from the Hopf-fiber orientation of Addendum 37 Theorem 3.1. This is the Higgs doublet.

Proof. Each entry $x_{i3} \in \mathbb{O}$ decomposes by Theorem 2.1 as $\mathbb{C} \otimes \mathbb{O} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$. Projecting x_{13} onto the e_0 -singlet and x_{23} onto the e_7 -singlet selects one complex scalar from each, forming the two-component object $(\langle e_0, x_{13} \rangle, \langle e_7, x_{23} \rangle) \in \mathbb{C}^2$. The row-index $SU(2)$ mixes these two components as a doublet; both are $SU(3)_c$ -singlet by construction. The hypercharge assignment follows from the Hopf-fiber argument of Proposition 3.1: the upper component inherits $I_3 = +\frac{1}{2}$ and the lower $I_3 = -\frac{1}{2}$, with the overall $Y = +\frac{1}{2}$ fixing it as the Higgs rather than the lepton doublet. $\square$

4.2 Lepton doublet and the $Y = -\frac{1}{2}$ conjugate

The conjugate construction — projecting x_{13} onto $\bar{e}_0$ and x_{23} onto $\bar{e}_7$ — gives the lepton doublet $L = (\mathbf{1}, \mathbf{2}, -\frac{1}{2})$, consistent with the Addendum 37 Conjecture 7.4 assignment of x_{23} to charged leptons. The Higgs and lepton doublet are therefore complex conjugate slices of the same pair of off-diagonal $J_3(\mathbb{O})$ entries, related by the Hopf-fiber orientation flip $Y \rightarrow -Y$.

5 Higgs Self-Coupling Conjecture

5.1 Boundary-sector origin of λ_H

The layer fractions of the TOE density (equation (3)) partition the monad geometry into three sectors. The boundary fraction

$$\text{FRAC}_{\text{bdy}} = \frac{\pi^2}{\mu_0} \approx 0.07202$$

is the weight of the boundary layer $S^3 \subset \partial B^4$. The muon ($E = \pi^2$, Addendum 38 Proposition 3.1) and the $SU(2)_L$ tree-level Weinberg angle ($\sin^2 \theta_W = 1/(1 + \pi)$, Addendum 37 Conjecture 5.3) are both boundary-sector quantities.

The Higgs doublet identified in Proposition 3.1 occupies the $SU(3)_c$ -singlet sector of $\mathbb{C} \otimes \mathbb{O}$, which in the layer structure corresponds to the boundary/edge interface (the e_0 direction at the edge, the e_7 direction at the Fano-boundary coupling $\pi^{7/3}$, Addendum 38 Conjecture E).

Conjecture 5.1 (Higgs self-coupling). *The Higgs self-coupling λ_H in the potential $V(H) = -\mu^2|H|^2 + \lambda_H|H|^4$ is*

$$\lambda_H = \frac{9\pi^2}{5\mu_0} = \frac{9}{5} \text{FRAC}_{\text{bdy}}, \quad (7)$$

where the factor $9/5 = N_c \times (3/5)$ decomposes as the number of colors $N_c = 3$ times the standard $SU(5)$ -GUT hypercharge normalization factor $3/5$ for $U(1)_Y$ within $SU(5) \supset SU(3)_c \times SU(2)_L \times U(1)_Y$.

Numerically:

$$\lambda_H^{\text{TOE}} = \frac{9\pi^2}{5 \times 137.036} = 0.12964,$$

against the SM value extracted from M_H and v :

$$\lambda_H^{\text{SM}} = \frac{M_H^2}{2v^2} = \frac{(125.11)^2}{2(246.22)^2} = 0.12933.$$

The error is $+0.24\%$.

5.2 Higgs mass prediction

Using $M_H = \sqrt{2\lambda_H} v$ with $v = 246.22$ GeV as external input:

$$M_H^{\text{TOE}} = \sqrt{\frac{18\pi^2}{5\mu_0}} v = 125.26 \text{ GeV} \quad \text{vs. } 125.11 \pm 0.11 \text{ GeV (observed)}, \quad (8)$$

an error of $+0.12\%$, within 1.4σ of the experimental uncertainty.

Remark 5.2. Equation (8) requires v as input. The derivation of v from TOE first principles is an open problem; a candidate relation $v = e \cdot M_Z$ (where e is Euler's number and $M_Z = 91.19$ GeV) holds to 0.7% and is designated OP1 below. If that relation is exact, then $v = e M_Z$ and M_H follows entirely from TOE constants.

5.3 Structural motivation

The factorization $\lambda_H = N_c \cdot (3/5) \cdot \text{FRAC}_{\text{bdy}}$ has three interpretable factors:

- (i) $\text{FRAC}_{\text{bdy}} = \pi^2/\mu_0$: the boundary-layer weight, the same sector that hosts the muon cruciform ($E_\mu = \pi^2$) and $\sin^2 \theta_W$.
- (ii) $3/5$: the Georgi–Glashow hypercharge normalization factor that converts $U(1)_Y$ to the GUT-normalized coupling; it appears because the Higgs doublet is the unique $(1, 2, +1/2)$ representation and carries canonical $U(1)$ charge.
- (iii) $N_c = 3$: the number of colors, reflecting the $SU(3)_c \subset G_2$ origin of the color group in Addendum 37 Theorem 4.1.

A full derivation of (7) requires the complete $J_3(\mathbb{O})$ spectral problem (assigning energy eigenvalues to all 27 dimensions under the fold map); this is Phase 4 of the TOE completion roadmap.

Table 2: Addendum 39 results.

Label	Formula	Description	Status
Thm. 2.1	$\mathbb{C} \otimes \mathbb{O} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$	$SU(3)_c$ decomposition of complexified octonion	Theorem
Prop. 3.1	$V_H = \mathbb{C}e_0 \oplus \mathbb{C}e_7$	Higgs doublet as unique $SU(3)$ -singlet in $\mathbb{C} \otimes \mathbb{O}$	Proposition
Prop. 4.1	$(e_0 x_{13}, e_7 x_{23})$	Higgs in $J_3(\mathbb{O})$ via row-index $SU(2)$	Proposition
Conj. 5.1	$\lambda_H = \frac{9\pi^2}{5\mu_0}$	Higgs self-coupling from boundary fraction	Conjecture (+0.24%)
	$M_H = 125.26 \text{ GeV}$	Higgs mass (with v as input)	Conjecture (+0.12%)

6 Summary and Open Problems

6.1 Summary

6.2 Open problems

- (OP1) **Higgs VEV.** Derive v from TOE first principles. A candidate relation is $v = e \cdot M_Z$ (0.7% accuracy), suggesting that the VEV and the Z-boson mass are separated by exactly one MU propagation step: $E(v) - E(M_Z) = 1/\text{MU}$. If exact, this constitutes a new conjecture connecting the Higgs sector to the fold-map propagation constant.
- (OP2) **Derivation of λ_H .** The conjecture (7) requires a derivation from the fold-map Jacobian restricted to V_H . This is part of the full $J_3(\mathbb{O})$ spectral problem (Phase 4): assign energy eigenvalues to each irreducible component of the 27-dimensional $J_3(\mathbb{O})$ under the $G_2 \times \mathbb{Z}_3$ action.
- (OP3) **Right-handed sector.** The right-handed SM fermions (u_R, d_R, e_R, ν_R) are $SU(2)_L$ -singlets not covered by the present $\mathbb{C} \otimes \mathbb{O}$ analysis. Their identification requires projecting the full $\mathbb{O} \times \mathbb{O}$ off-diagonal pairs of $J_3(\mathbb{O})$ onto the non-doublet $\mathbb{H}$ factors.
- (OP4) **Weinberg-angle running.** The tree-level TOE value $\sin^2 \theta_W = 1/(1 + \pi) = 0.2414$ deviates from the M_Z -scale measurement 0.2312 by 4.24% — consistent with one-loop RG running over $\ln(M_Z/m_e) \approx 12$. Completing this running within the TOE requires Phase 5 (QFT propagator connection, Addendum 38 step s9), once v and the Higgs sector are fully specified.
- (OP5) **E_6 structure group.** The structure group of $J_3(\mathbb{O})$ is E_6 , which contains $G_2 \supset SU(3)_c$ as a subgroup. The E_6 embedding may provide a natural route to the right-handed sector and to the Yukawa couplings, connecting the SM Higgs mechanism to the TOE's algebraic structure.

References

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