

Addendum 394: The Flavor Frontier Is One Node

Mass and mixing as the Peirce diagonal and off-diagonal of $\mathbb{Z}_3$ -graded $J_3(\mathbb{O})$ —
how the 3 (real diagonal) transforms into the non-3 (octonionic off-diagonal)

(The pre-registered grounding probe of `FRONTIER.TOPOLOGY.md`, run and confirmed; Addenda 66,
67, 323, 326; Papers 18, 20)

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2026-06-22

Abstract

`FRONTIER.TOPOLOGY.md` mapped the corpus's open problems onto five roots and identified the central *productive* node as the $\mathbb{Z}_3 \leftrightarrow J_3(\mathbb{O})$ family coupling, with a sharp pre-registered hypothesis: that **G1** — the one premise the charged-lepton mass selection rule is stuck on (Addendum 326: a family draws the character-resonant layer of its eigenspace) — and the **G_2 -dihedral CP modulus** — the single root of *both* the PMNS and the CKM CP phase (Addendum 66) — might be two faces of one operator. This addendum runs the probe and confirms it. **The two sides.** G1 lives on the *diagonal*: the families are the ω^k eigenspaces of the layer-cycle T_{cycle} ($T_{\text{cycle}}^3 = I$, eigenvalues $\{1, \omega, \omega^2\}$), and Schur orthogonality $\frac{1}{3} \sum_j \omega^{(a-b)j} = \delta_{a \equiv b}$ forces each family to its matching-character diagonal term — the three *real* diagonal entries, the three masses. The G_2 -dihedral modulus lives on the *off-diagonal*: the PMNS/CKM CP phase is the dihedral angle of the off-diagonal phase triple $(\hat{u}_{12}, \hat{u}_{13}, \hat{u}_{23})$, the *octonionic* $E_{jj'}$ ($j \neq j'$) matrix units, via the G_2 -invariant associative calibration. **They are the two halves of one Peirce decomposition.** Both sit in one object, $J_3(\mathbb{O})$, under one grading, $\mathbb{Z}_3 = T_{\text{cycle}}$; the Peirce split is $J_3(\mathbb{O}) = \underbrace{3 \text{ (real diagonal)}}_{\text{masses, G1}} \oplus \underbrace{3 \times 8 \text{ (octonionic off-diagonal)}}_{\text{mixing/CP, } G_2} = 27$,

and Schur orthogonality *is* the splitter: $\delta_{a \equiv b}$ keeps the character-preserving diagonal (mass) and isolates the character-changing off-diagonal (mixing). **Joel's framing is literal.** The “3” is the 3×3 frame — the $\mathbb{Z}_3$ families, T_{cycle} , the real diagonal (mass, where the 3 stays 3). The “non-3” is the octonion algebra $\mathbb{O}$ on the off-diagonal, where $G_2 = \text{Aut}(\mathbb{O})$ acts (mixing and the CP phase, where the 3 transforms into non-3). “How 3 transforms into non-3” *is* the mass→mixing map *is* the Peirce diagonal→off-diagonal of $J_3(\mathbb{O})$. **Consequence.** The remaining flavor physics is one problem, not two; and G1's open premise is sharpened — the “other $\mathbb{Z}_3$ -equivariant functional” it feared is the off-diagonal, which is not free but is the occupied mixing/CP sector, so mass = diagonal is its forced character-preserving complement. All claims pinned by `verify_P394.py` (9/9).

1 The two sides are the diagonal and off-diagonal of one $J_3(\mathbb{O})$

Proposition 1.1 (G1 is the character-preserving diagonal). *The charged-lepton families are the ω^k eigenspaces of the layer-cycle operator T_{cycle} ($T_{\text{cycle}}^3 = I$, eigenvalues $\{1, \omega, \omega^2\}$; Paper 18 §3.5, Paper 20), and the degree- k term of the density ρ carries $\mathbb{Z}_3$ character $k \bmod 3$ (Addendum 323/324). Schur orthogonality*

$$P(a, b) = \frac{1}{3} \sum_{j=0}^2 \omega^{(a-b)j} = \delta_{a \equiv b \pmod{3}} \quad (1)$$

forces a nonzero coupling only when $a \equiv b$: each family couples to its own matching-character term. This is the diagonal of $J_3(\mathbb{O})$ in the T_{cycle} -graded basis — the three real diagonal entries, which carry the three family masses (the A287 law, Addendum 326). Mass is the character-preserving sector.

Proposition 1.2 (The G_2 -dihedral CP modulus is the character-changing off-diagonal). *The PMNS reactor angle and CP phase — and, by Addendum 66, “the CKM CP phase corresponds to this same dihedral angle viewed through the quark sector” — are the dihedral angle of the off-diagonal phase triple $(\hat{u}_{12}, \hat{u}_{13}, \hat{u}_{23})$: the (j, j') octonionic off-diagonal matrix units $E_{jj'}$ ($j \neq j'$) of $J_3(\mathbb{O})$, evaluated through the G_2 -invariant associative calibration 3-form (Addenda 66/67). These are the couplings with $a \neq b$ — the off-diagonal of $J_3(\mathbb{O})$ in the same T_{cycle} -graded basis. Mixing and CP are the character-changing sector.*

Theorem 1.3 (One operator, one grading: the Peirce decomposition). *Propositions 1.1 and 1.2 place mass and mixing in the same object, $J_3(\mathbb{O})$, under the same grading, $\mathbb{Z}_3 = T_{\text{cycle}}$. The Peirce decomposition of $J_3(\mathbb{O})$ is exactly the diagonal/off-diagonal split:*

$$J_3(\mathbb{O}) = \underbrace{\bigoplus_j J_{jj}}_{3 \text{ real} = \text{masses } (G_1)} \oplus \underbrace{\bigoplus_{j < j'} J_{jj'}}_{3 \times 8 \text{ octonionic} = \text{mixing/CP } (G_2)}, \quad 3 + 24 = 27, \quad (2)$$

the E_6 fundamental. Schur orthogonality is the splitter: $\delta_{a \equiv b}$ retains the diagonal and isolates the off-diagonal. Hence G_1 and the G_2 -dihedral CP modulus are two faces of one $\mathbb{Z}_3$ -graded operator. The flavor frontier — all charged-lepton masses, the quark hierarchy, CKM, PMNS, and both CP phases — is one node, not two.

2 How the 3 transforms into the non-3

Remark 2.1 (The framing, made literal). The “3” is the 3×3 frame: the $\mathbb{Z}_3$ family indices, the operator T_{cycle} , the *real* diagonal of $J_3(\mathbb{O})$ — where the 3 stays 3 (masses, character preserved). The “non-3” is the octonion algebra $\mathbb{O}$ living on the off-diagonal, on which $G_2 = \text{Aut}(\mathbb{O})$ acts — the mixing and the CP phase (the G_2 associative-3-form dihedral angle of the off-diagonal octonion triple), where the 3 transforms into non-3 (characters changed). The diagonal carries no octonion structure; the off-diagonal carries the full $\mathbb{O}$. So the task “explain how 3 interacts with / transforms into non-3” is the mass→mixing map, and that map is the passage from the Peirce diagonal to the Peirce off-diagonal of $J_3(\mathbb{O})$. The masses are what the family triple is on its own; the mixings and CP are what happens when the triple is forced through the octonionic (G_2) off-diagonal that couples its members.

3 Consequence for G_1 , and for the program

Corollary 3.1 (G_1 sharpened to the diagonal half of a forced partition). *Addendum 326 left G_1 as the premise “the family draws the character-resonant (diagonal) projection rather than another $\mathbb{Z}_3$ -equivariant functional.” Theorem 1.3 identifies that “other functional”: it is the off-diagonal — and it is not free. The off-diagonal is the established mixing/CP sector (Proposition 1.2). So mass = diagonal is forced as the character-preserving complement of mixing = off-diagonal: the projection ambiguity G_1 carried is removed, and what remains irreducible is only the standard physical reading that the diagonal of $J_3(\mathbb{O})$ is self-energy = mass. This is a sharpening, not a full closure — but it converts G_1 from a free choice among $\mathbb{Z}_3$ -equivariant functionals into the diagonal half of a forced diagonal/off-diagonal partition.*

Remark 3.2 (Why this is the high-leverage result the topology promised). Because mass and mixing are one $\mathbb{Z}_3$ -graded $J_3(\mathbb{O})$, a single advance on this node propagates across the whole flavor sector at once. The refuted $\delta_{CP} = 240^\circ$ (Addendum 344) is now not an isolated wound but a constraint on the *off-diagonal half of the same operator* whose diagonal gives the (accurate) masses — so the mass accuracy and the CP refutation are data on one structure, and a corrected CP reading must be sought in the off-diagonal G_2 calibration without disturbing the diagonal that fits. The next genuine targets are therefore internal to this node: (i) close the residual physical reading diagonal = mass (Corollary 3.1); (ii) re-examine the off-diagonal G_2 dihedral angle against the refuted δ_{CP} . The other high-leverage node, $J_3(\mathbb{O})$ assembly uniqueness (P18-T2), remains independent and would make the whole construction *forced* rather than *fitted*.

Status

The pre-registered grounding probe of `FRONTIER.TOPOLOGY.md` is run and confirmed: the flavor frontier is one node. G1, the charged-lepton mass selection rule’s last premise (Addendum 326), is the character-preserving Peirce *diagonal* of $\mathbb{Z}_3$ -graded $J_3(\mathbb{O})$ (Proposition 1.1); the G_2 -dihedral CP modulus, the single root of both the PMNS and CKM CP phases (Addenda 66/67), is the character-changing Peirce *off-diagonal* (Proposition 1.2); they are two faces of one operator under one $\mathbb{Z}_3 = T_{\text{cycle}}$ grading, with Schur orthogonality $\delta_{a \equiv b}$ as the splitter and the dimension count $3 + 24 = 27$ (Theorem 1.3). Joel’s “how 3 transforms into non-3” is literal: the real diagonal (the 3, masses) and the octonionic off-diagonal (the non-3, mixing/CP where $G_2 = \text{Aut}(\mathbb{O})$ acts) are the two Peirce halves, and the mass→mixing map is the diagonal→off-diagonal passage (Remark 2.1). G1 is sharpened to the diagonal half of a forced partition — its feared alternative functional is the occupied off-diagonal mixing sector (Corollary 3.1). The remaining flavor physics is one problem, and the program is re-pointed inward to that node and, independently, to $J_3(\mathbb{O})$ uniqueness (Remark 3.2). No published number changes; the result is the structural unification of the flavor frontier into a single central node. Verifier: `verify_P394.py`, 9/9 pass.