

Addendum 390: The Dihedral Family Group

$D_3 = S_3 =$ the A_2 Weyl group unifies families, C/P , and the void —
but its reflection deficits overshoot H_0
(Extends Addenda 385, 386, 387, 389; Papers 18, 06)

Léon Fernando Vlegels

2026-06-19

Abstract

Addendum 389 found the family quotient $L(3,1) = S^3/\mathbb{Z}_3$ to be a *free* action (smooth, the horizon factor $F = 2\pi$, no conical deficit), making the H_0 over-prediction structural. But $\mathbb{Z}_3$ is only the rotation subgroup: Addenda 385/386 showed the corpus also carries a $\mathbb{Z}_2$ reflection — the conjugation $\omega \leftrightarrow \bar{\omega}$ that realises *both* C and P . The rotations and that reflection together generate the dihedral group. **The structure.** With $r = T_{\text{cycle}}$ (the $\mathbb{Z}_3$ family generator) and s the $C = P$ reflection, the relations $r^3 = s^2 = 1$, $srs^{-1} = r^{-1}$ generate D_3 (order 6) $= S_3 =$ the Weyl group of $A_2/\text{SU}(3)$. So the family/CPT discrete symmetry is D_3 : the three rotations are the three fermion families (free, 389), and the three reflections are $C = P$ (with $CP = s^2 = \text{id}$, recovering Addendum 386). The reflections' fixed locus is the real/antipodal locus — the void at $\Theta = \pi$ (Addendum 387), which is dark energy. So D_3 unifies four threads in one group: the families, C and P , the void / dark energy, and the $A_2/\text{SU}(3)$ Weyl group. **It corrects 389:** $S^3/\mathbb{Z}_3$ is free, but the *full* S^3/D_3 is not — the reflections fix the void, so it is an orbifold; 389's smoothness held only for the rotations. **But it does not rescue H_0 .** The data want $F \in [5.587, 6.066]$ (a 3–11% deficit); the reflection orbifold deficits are far larger — a $\mathbb{Z}_2$ cone (deficit π) gives $F = \pi$, $H_0 = 37.9$; the D_3 fundamental domain gives $F = 2\pi/3$, $H_0 = 25.3$ — both far below the window. The available deficits are quantised too coarsely (0 or $\sim \pi$), with nothing in the 3–11% range. So D_3 deepens rather than rescues 389's negative: the H_0 tension stays structural. A structural win and an honest H_0 negative. All claims pinned by `verify_P390.py` (9/9).

1 The family/CPT group is dihedral

Theorem 1.1 ($D_3 = S_3 =$ the A_2 Weyl group). *Let $r = T_{\text{cycle}}$ be the $\mathbb{Z}_3$ family generator (Paper 18; the three families, Paper 06) and s the reflection $\omega \mapsto \omega^2 = \bar{\omega}$ that realises $C = P$ (Addenda 385/386). Then $r^3 = s^2 = 1$ and $srs^{-1} = r^{-1}$, so $\langle r, s \rangle = D_3$, the dihedral group of order 6, which is S_3 and the Weyl group of $A_2/\text{SU}(3)$. The three rotations are the three families; the three reflections are $C = P$. Since $C = P = s$, the composite $CP = s^2 = \text{id}$, recovering Addendum 386's $CP = \text{id}$ from the group structure: C and P are the same reflection, so their product is the identity.*

Corollary 1.2 (D_3 unifies four threads). *The dihedral group ties together what had been separate results. The three families are its rotations ($\mathbb{Z}_3$). C and P are its reflections (Addenda 385/386). The reflections' fixed locus — the real/antipodal locus, $\omega = \pm 1$, the void at $\Theta = \pi$ — is dark energy (Addendum 387). And $D_3 \cong S_3$ is the Weyl group of $A_2/\text{SU}(3)$, the corpus's gauge-family structure. Families, C/P , the void, and the Weyl group are one group.*

2 The correction to 389, and the H_0 verdict

Proposition 2.1 (S^3/D_3 is an orbifold). *Addendum 389's claim that $S^3/\mathbb{Z}_3 = L(3,1)$ is a free action (hence smooth, $F = 2\pi$) is true of the rotation subgroup only. The full S^3/D_3 is not free: the reflections fix a locus (the real/antipodal locus, the void at $\Theta = \pi$), so S^3/D_3 is an orbifold with a singular locus there. So 389's smoothness was incomplete — it covered the families, not the full family/CPT group.*

Theorem 2.2 (The reflection deficits overshoot H_0). *Although S^3/D_3 has a singular locus (Proposition 2.1), the conical deficits it provides are far too large for the H_0 window. The data want $F \in [5.587, 6.066]$ (a 3–11% deficit below 2π). A $\mathbb{Z}_2$ reflection cone has deficit π , giving $F = \pi$ and $H_0 = 37.9$; the D_3 fundamental domain on the circle gives $F = 2\pi/3$ and $H_0 = 25.3$. Both lie far below the data (67–73). The available structural deficits are quantised too coarsely — 0 (the $\mathbb{Z}_3$ -free case, $F = 2\pi$, $H_0 = 75.8$) or $\sim\pi$ (a reflection cone, $H_0 \sim 38$) — with nothing in the 3–11% range. The dihedral does not fine-tune F ; it jumps.*

Remark 2.3 (A structural win, an honest H_0 negative). So the dihedral both deepens and confirms Addendum 389. It deepens it: S^3/D_3 is an orbifold, not the smooth space 389 described, and the corpus's discrete family/CPT symmetry is richer (D_3 , not $\mathbb{Z}_3$) and more unified (Corollary 1.2) than had been recorded. It confirms it: the orbifold structure available overshoots the H_0 window rather than fine-tuning it (Theorem 2.2), so $H_0 = 75.8$ and the $\sim 3\sigma$ tension remain structural. The dihedral's value is the structural unification of families, C/P , the void, and the A_2 Weyl group — not an H_0 rescue. The H_0 fix, if any, still lies deeper: the $a_0 = cH_0/2\pi$ relation, the chart anchoring, or the closure hypothesis (Addendum 370).

Status

The family/CPT discrete group is the dihedral $D_3 = S_3 =$ the $A_2/\text{SU}(3)$ Weyl group: the three rotations are the three fermion families ($\mathbb{Z}_3$, T_{cycle}) and the three reflections are $C = P$ (Addenda 385/386), with $CP = s^2 = \text{id}$ recovered from the group (Theorem 1.1). The reflections' fixed locus is the void / dark energy (Addendum 387), so D_3 unifies families, C/P , the void, and the Weyl group in one structure (Corollary 1.2). This corrects Addendum 389: $S^3/\mathbb{Z}_3$ is free, but the full S^3/D_3 is an orbifold (Proposition 2.1). But the reflection orbifold deficits overshoot the H_0 window — $F = \pi$ gives $H_0 = 37.9$, $F = 2\pi/3$ gives $H_0 = 25.3$, both far below the data — with no 3–11% deficit available (Theorem 2.2). So D_3 is a structural win and an honest H_0 negative (Remark 2.3): the tension stays structural and the fix lies deeper. No published number changes. Verifier: `verify_P390.py`, 9/9 pass.