

Quark Sector and Neutrino Structure from $J_3(\mathbb{O})$ Geometry

Addendum 38 to the TOE Corpus

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Abstract

We extend the TOE corpus mass derivations to the quark sector and neutrino structure, using the geometry of $J_3(\mathbb{O})$ (the Albert algebra) established in Addendum 37. The mass formula $m/m_e = \exp(\text{MU} \cdot (E - \pi))$ with $\text{MU} = \mu_1/\mu_0 = 0.7933$ and quark energies derived from PDG masses reveals three structural relationships. **Conjecture A:** the charm quark energy $E_c = 12.996$ coincides with the non-bulk sector sum $\mu_0^{\text{sub}} = \pi^2 + \pi = 13.011$ (difference -0.015), predicting $m_c = 1285$ MeV against PDG 1270 MeV ($+1.2\%$). **Conjecture B:** the top/charm mass ratio $m_t/m_c = 135.98 \approx \mu_0 = \alpha^{-1} = 137.036$ (error -0.77%), arising because the energy gap $E_t - E_c = \ln(\mu_0)/\text{MU}$. **Conjecture C:** the Cabibbo angle $\theta_C = \pi/\dim(G_2) = \pi/14$, giving $\sin(\pi/14) = 0.22252$ against PDG $|V_{us}| = 0.22431$ (error -0.80%). For the neutrino sector, the $\mathbb{Z}_3$ symmetry of $J_3(\mathbb{O})$ gives tribimaximal mixing as the exact fixed point, in agreement with $\sin^2 \theta_{12}$ and $\sin^2 \theta_{23}$ to $\sim 8\%$. A Phase 1.5 analysis proves that the muon energy $E_\mu = \pi^2$ lies entirely below the self-lensing range $[13.177, 13.617]$ and cannot be derived from an S^3/S^1 sub-fold balance; deriving m_μ requires a new fold formalism extending Paper 04.

1 Introduction

1.1 Corpus context

The TOE is grounded in the monad density

$$\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x, \quad x \in [0, 1], \quad (1)$$

whose integral gives $\mu_0 = 4\pi^3 + \pi^2 + \pi = \alpha^{-1} = 137.036$, and whose first moment gives $\mu_1 = 16\pi^3/5 + 3\pi^2/4 + 2\pi/3 = 108.717$. The moment ratio $\text{MU} = \mu_1/\mu_0 = 0.7933$ enters every mass formula.

The charged lepton mass sector is now fully derived: the electron energy $E_e = \pi$ is the edge layer integral, the muon energy $E_\mu = \pi^2$ is the boundary layer integral, and the tau mass was derived in Addendum 36 via the self-lensing fold-map at the B^4/S^3 boundary, with $\lambda_\tau = \mu_1/(\mu_0 + \mu_1) = 0.4424$ and $E_\tau = 13.419$. Addendum 37 derived the Standard Model gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ from the division algebra tower $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$ and established $J_3(\mathbb{O})$ as the home of three generations and color degrees of freedom.

The present paper addresses four open items left by Addendum 37:

- the muon sub-fold conjecture (Phase 1.5, §2);
- neutrino mixing from $\mathbb{Z}_3$ and absolute mass scale (Phase 3, §3);
- quark mass relationships from sector geometry (Phase 4, §4–§6).

1.2 Constants and notation

We use throughout:

$$\mu_0 = 4\pi^3 + \pi^2 + \pi = 137.036, \quad (2)$$

$$\mu_1 = \frac{16\pi^3}{5} + \frac{3\pi^2}{4} + \frac{2\pi}{3} = 108.717, \quad (3)$$

$$\text{MU} = \mu_1/\mu_0 = 0.7933, \quad (4)$$

$$\mu_0^{\text{sub}} = \pi^2 + \pi = 13.011, \quad (5)$$

$$\mu_1^{\text{sub}} = \frac{3\pi^2}{4} + \frac{2\pi}{3} = 9.497. \quad (6)$$

The quantity μ_0^{sub} is the integral of the non-bulk sector $3\pi^2 x^2 + 2\pi x$ over $[0, 1]$ (boundary + edge layers only). The quantity μ_1^{sub} is its first moment.

The quark energy for a quark of mass m is defined by the same mass formula as for the charged leptons:

$$E_q = \pi + \frac{\ln(m_q/m_e)}{\text{MU}}, \quad m_e = 0.51099895 \text{ MeV}. \quad (7)$$

All quark masses are PDG 2024 values (MS-bar at 2 GeV, except the top quark at the pole mass m_t).

1.3 Summary of results

Result	Status	Ref	Error
Muon sub-fold failure	Proven dead-end	Prop. 2.1	structural gap
Neutrino oscillation from $\mathbb{Z}_3$	Consistent	Prop. 3.1	$\sim 8\%$
Absolute neutrino mass (Type 1 gap)	Open	Prop. 3.3	—
Charm at μ_0^{sub}	Conjecture A	Conj. 4.1	+1.2%
$m_t/m_c = \alpha^{-1}$	Conjecture B	Conj. 5.1	−0.77%
Cabibbo = $\pi/14$	Conjecture C	Conj. 6.1	−0.80%

2 Phase 1.5: Muon Sub-Fold Analysis

2.1 Setup

Addendum 37 Remark ?? (carried over from Addendum 36) proposed that the muon mass might follow from a fold-map balance on the S^3/S^1 sub-boundary, analogous to how the tau mass follows from the B^4/S^3 fold. We prove here that this approach fails.

The self-lensing energy function (Paper 04 §7, Addendum 36 §8) is

$$E_{\text{self}}(\lambda) = (1 - \lambda) E_{\text{low}} + \lambda E_{\text{high}}, \quad \lambda \in [0, 1], \quad (8)$$

where the endpoints bracket the self-lensing range. For the full B^4/S^3 fold: $E_{\text{low}} = 13.177$, $E_{\text{high}} = 13.617$, so $E_{\text{self}} \in [13.177, 13.617]$ for all $\lambda \in [0, 1]$.

The tau derivation uses the fixed-point weight

$$\lambda_\tau = \frac{\mu_1}{\mu_0 + \mu_1} = \frac{108.717}{137.036 + 108.717} = 0.4424, \quad (9)$$

yielding $E_{\text{self}}(\lambda_\tau) = 13.419$, which reproduces $m_\tau/m_e = \exp(\text{MU} \cdot (13.419 - \pi))$ to $< 0.04\%$.

2.2 Sub-fold balance candidate

Define the S^3/S^1 sub-fold weight by analogy, replacing (μ_0, μ_1) with $(\mu_0^{\text{sub}}, \mu_1^{\text{sub}})$:

$$\lambda_{\text{sub}} = \frac{\mu_1^{\text{sub}}}{\mu_0^{\text{sub}} + \mu_1^{\text{sub}}} = \frac{9.497}{13.011 + 9.497} = \frac{9.497}{22.508} = 0.422. \quad (10)$$

Proposition 2.1 (Sub-fold failure). *The muon energy $E_\mu = \pi^2 \approx 9.870$ cannot be derived from the sub-fold balance condition $E_{\text{self}}(\lambda_{\text{sub}})$ applied to the S^3/S^1 boundary.*

Proof. The self-lensing function E_{self} maps $[0, 1]$ into the range $[13.177, 13.617]$ (Paper 04 §7). This range is entirely above $E_\mu = \pi^2 \approx 9.870$. Evaluating at the sub-fold weight $\lambda_{\text{sub}} = 0.422$ gives

$$E_{\text{self}}(0.422) = (1 - 0.422) \times 13.177 + 0.422 \times 13.617 = 13.36, \quad (11)$$

which is in the tau-mass region, not the muon-mass region. Concretely:

$$\frac{m_{\text{pred}}}{m_e} = \exp(\text{MU} \cdot (13.36 - \pi)) = \exp(0.7933 \times 10.22) = \exp(8.106) \approx 3300, \quad (12)$$

whereas $m_\mu/m_e = 206.77$. The predicted mass is ~ 1690 MeV, of order the tau-mass scale ($m_\tau/m_e = 3477$), not the muon scale.

No choice of $\lambda \in [0, 1]$ can produce $E_{\text{self}}(\lambda) = 9.870$ because $9.870 < 13.177 = \min E_{\text{self}}$. The self-lensing mechanism is inoperative at the muon energy. $\square$

Remark 2.2 (Why the approach fails structurally). The muon energy π^2 lies in the *boundary layer sector* of $\rho(x)$, at $\int_0^1 3\pi^2 x^2 dx = \pi^2$. Self-lensing in Paper 04 operates at the B^4/S^3 boundary by balancing bulk and boundary energies, which are both above 4π . The boundary-to-edge sub-fold operates at S^3/S^1 , where one sector energy is π^2 and the other is π — a ratio of π , not of order 1. No self-lensing in this sub-fold can produce an energy as low as π^2 from above; the mechanism would need to work *from below*, requiring a new formalism.

Remark 2.3 (Open problem: muon from S^3/S^1 fold). The muon mass problem requires extending the Paper 04 fold-map to the S^3/S^1 sub-boundary. The fixed-point condition

$$\lambda_\mu = \frac{E_\mu}{E_e + E_\mu} = \frac{\pi^2}{\pi + \pi^2} = \frac{\pi}{1 + \pi} \approx 0.759 \quad (13)$$

(Addendum 37 Remark ??) is the natural candidate for such a fixed point, but the function $E_{\text{self}}^{(S^3)}(\lambda)$ for the boundary-to-edge fold does not exist in the current corpus. Its derivation, starting from the sub-density $3\pi^2 x^2 + 2\pi x$ on S^3/S^1 , is designated **Phase 1.5** of the TOE completion roadmap. The numerical target is:

$$\frac{m_\mu}{m_e} = 206.768 \iff E_{\text{self}}^{(S^3)}(0.759) = \pi^2 + \frac{\ln(206.768)}{\text{MU}} = \pi^2 + 6.742 = 16.611, \quad (14)$$

which is *above* the lepton self-lensing range — consistent with a new fold mechanism rather than the existing B^4/S^3 one.

3 Neutrino Sector

3.1 Oscillation from $\mathbb{Z}_3$ symmetry

Paper 28 Proposition 6.2 gives the exact two-flavor oscillation formula:

$$P(\nu_e \rightarrow \nu_e; L) = \frac{1 + v(q_{\text{rel}}(L))}{2}, \quad (15)$$

where v is the Bloch-vector magnitude of the relative state $q_{\text{rel}}(L)$ evolving on S^3 as a function of propagation length L .

The three-generation case requires three families, provided by the $\mathbb{Z}_3$ permutation symmetry of the $J_3(\mathbb{O})$ diagonal (Addendum 37 Proposition ??) via $\pi_1(L(3, 1)) = \mathbb{Z}_3$ (Paper 06).

Proposition 3.1 (Tribimaximal mixing as $\mathbb{Z}_3$ fixed point). *The $\mathbb{Z}_3$ -symmetric fixed point of the neutrino mixing matrix, derived from the cyclic permutation symmetry $\alpha_1 \leftrightarrow \alpha_2 \leftrightarrow \alpha_3$ of the $J_3(\mathbb{O})$ diagonal, is the tribimaximal (TBM) mixing matrix with*

$$\sin^2 \theta_{12} = \frac{1}{3}, \quad \sin^2 \theta_{23} = \frac{1}{2}, \quad \theta_{13} = 0. \quad (16)$$

Proof. The $\mathbb{Z}_3$ cyclic permutation $T : \alpha_i \mapsto \alpha_{i+1 \bmod 3}$ generates a symmetry group $\langle T \rangle \cong \mathbb{Z}_3$ acting on the flavor basis of $J_3(\mathbb{O})$. The unique PMNS matrix invariant under this cyclic permutation (up to rephasing) is the tribimaximal matrix:

$$U_{\text{TBM}} = \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ -\sqrt{1/6} & \sqrt{1/3} & -\sqrt{1/2} \\ -\sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \end{pmatrix}. \quad (17)$$

Reading off the mixing angles: $|U_{e2}|^2 = 1/3$ gives $\sin^2 \theta_{12} = 1/3$; $|U_{\mu 3}|^2 = 1/2$ gives $\sin^2 \theta_{23} = 1/2$; $|U_{e3}|^2 = 0$ gives $\theta_{13} = 0$. The derivation is equivalent to the standard group-theoretic construction of TBM from $S_4 \supset \mathbb{Z}_3 \times \mathbb{Z}_2$; the $\mathbb{Z}_3$ here is that of $L(3,1)$. See [4] for the TBM matrix; the $\mathbb{Z}_3$ identification with $J_3(\mathbb{O})$ is as in Addendum 37 Proposition ??.

Remark 3.2 (Comparison with PDG 2024). The PDG values (NuFIT 5.3, normal ordering) are:

Angle	PDG	TBM	Difference	Status
$\sin^2 \theta_{12}$	0.307	$1/3 = 0.333$	-8%	TBM overestimates
$\sin^2 \theta_{23}$	0.545	$1/2 = 0.500$	$+9\%$	TBM underestimates
$\sin^2 \theta_{13}$	0.0218	0	off by ~ 0.022	TBM predicts zero

The $\sim 8\%$ deviations from TBM are of the order expected from off-diagonal $J_3(\mathbb{O})$ phases not yet fixed by the $\mathbb{Z}_3$ constraint alone. In particular, $\theta_{13} \neq 0$ (the “reactor angle”) requires a nonzero CP-violating phase δ that presumably arises from the octonion phase structure of $x_{ij} \in \mathbb{O}$ in the off-diagonal entries of $J_3(\mathbb{O})$. This constitutes an open problem requiring a full representation-theoretic treatment of the G_2 action on the off-diagonal elements.

3.2 Absolute neutrino mass scale

Proposition 3.3 (Type 1 gap: absolute mass scale). *The absolute neutrino mass scale is not constrained by the density $\rho(x)$ alone.*

Proof. Applying the mass formula (7) to a representative neutrino mass $m_\nu \sim 0.05$ eV (consistent with oscillation data):

$$E_\nu = \pi + \frac{\ln(0.05 \text{ eV}/0.511 \text{ MeV})}{\text{MU}} = \pi + \frac{\ln(9.78 \times 10^{-8})}{0.7933} = \pi + \frac{-16.14}{0.7933} = 3.1416 - 20.35 = -17.2. \quad (18)$$

The result $E_\nu \approx -17$ is far outside the domain $[0, \infty)$ of the self-lensing function E_{self} and below the floor $E_e = \pi \approx 3.14$ set by the edge layer. The formula predicts no quanta in the density $\rho(x)$ at such an energy — the neutrino mass is not encoded in the monad density.

This is a **Type 1 gap**: the formula breaks down because the relevant physics (the Higgs mechanism and/or the seesaw mechanism) is not yet incorporated in $\rho(x)$. The Yukawa coupling of neutrinos to the Higgs field would require a separate sector of the operator $\hat{O}$ (Paper 18) not treated in the current corpus. $\square$

Remark 3.4 (Neutrino mass differences vs. absolute scale). Although the absolute scale is a Type 1 gap, the oscillation mass *differences* Δm_{21}^2 and Δm_{31}^2 may yet be derivable from the $J_3(\mathbb{O})$ off-diagonal octonion phases x_{ij} . The off-diagonal entries carry 8 real degrees of freedom each; the three off-diagonal entries together carry 24 DOF that are constrained only by $G_2 \times \mathbb{Z}_3$ symmetry. Identifying the mass-squared differences with geometric invariants of this action is an open problem, designated **Phase 3.5** of the completion roadmap.

4 Charm Quark at the Non-Bulk Sector Sum

4.1 Quark energies from PDG masses

Using the mass formula (7) with PDG 2024 values:

Quark	PDG mass	Energy E_q
u	2.16 MeV	$E_u = \pi + \ln(2.16/0.511)/0.7933 = 4.959$
d	4.67 MeV	$E_d = \pi + \ln(4.67/0.511)/0.7933 = 5.931$
s	93.4 MeV	$E_s = \pi + \ln(93.4/0.511)/0.7933 = 9.701$
c	1270 MeV	$E_c = \pi + \ln(1270/0.511)/0.7933 = 12.996$
b	4180 MeV	$E_b = \pi + \ln(4180/0.511)/0.7933 = 14.498$
t	172,690 MeV	$E_t = \pi + \ln(172690/0.511)/0.7933 = 19.187$

4.2 The charm–boundary coincidence

Conjecture 4.1 (Charm at non-bulk sector sum). *The charm quark energy equals the non-bulk sector integral $\mu_0^{\text{sub}} = \pi^2 + \pi$:*

$$E_c = \mu_0^{\text{sub}} = \pi^2 + \pi = 13.011, \quad (19)$$

up to a deviation of $\delta E_c = E_c^{\text{PDG}} - \mu_0^{\text{sub}} = 12.996 - 13.011 = -0.015$.

The predicted charm mass is

$$m_c^{\text{pred}} = m_e \cdot \exp(\text{MU} \cdot (\mu_0^{\text{sub}} - \pi)) = m_e \cdot \exp(\text{MU} \cdot \pi^2). \quad (20)$$

Numerically: $\text{MU} \cdot \pi^2 = 0.7933 \times 9.8696 = 7.828$, so $m_c^{\text{pred}} = 0.511 \times e^{7.828} = 0.511 \times 2513 = 1285 \text{ MeV}$. The PDG value is 1270 MeV; the error is +1.2%.

Remark 4.2 (Physical interpretation). The non-bulk integral $\mu_0^{\text{sub}} = \pi^2 + \pi$ is the total monad density in the boundary and edge sectors, i.e., the total measure of the S^3 boundary plus the S^1 fiber. The charm quark sitting at this energy means it is the *quark-sector analog of the electron*: the lowest-mass quark at the interface between the boundary and edge layers. In the lepton sector, the electron energy $E_e = \pi$ is the edge integral alone; the charm energy $E_c \approx \pi^2 + \pi$ is the combined non-bulk integral.

This is structurally consistent with the $J_3(\mathbb{O})$ assignment in Addendum 37 Conjecture ??: the charm quark belongs to the up-type doublet and is the lowest-mass quark that can be produced in electroweak interactions, corresponding to the boundary/edge sub-system.

Remark 4.3 (Closure condition). Conjecture 4.1 requires showing that the charm sector energy is fixed by a balance condition on the $S^3 \cup S^1$ sub-manifold, analogous to how $E_e = \int_0^1 2\pi x dx = \pi$ is fixed by the edge layer. The identity $\mu_0^{\text{sub}} = \int_0^1 (3\pi^2 x^2 + 2\pi x) dx$ is exact; the physical argument is that the charm is the lightest quark for which color confinement operates in the boundary/edge regime rather than the bulk. A proof requires the complete quark-sector treatment of $J_3(\mathbb{O})$ under $SU(3)_c$ (Addendum 37 Conjecture ??).

5 Top/Charm Mass Ratio and α^{-1}

5.1 Derivation

Conjecture 5.1 (Top/charm mass ratio = α^{-1}). *The top-to-charm mass ratio equals the inverse fine-structure constant:*

$$\frac{m_t}{m_c} = \mu_0 = \alpha^{-1} = 137.036. \quad (21)$$

Equivalently, the top/charm energy gap satisfies

$$E_t - E_c = \frac{\ln \mu_0}{\text{MU}}. \quad (22)$$

Derivation. From the mass formula, $m_t/m_c = \exp(\text{MU} \cdot (E_t - E_c))$. Setting this equal to μ_0 gives $\text{MU} \cdot (E_t - E_c) = \ln \mu_0$, i.e. $E_t - E_c = \ln(137.036)/0.7933 = 4.9196/0.7933 = 6.202$.

Numerically from PDG masses: $E_t - E_c = 19.187 - 12.996 = 6.191$. Difference from the predicted gap: $6.202 - 6.191 = 0.011$.

The mass ratio: $m_t/m_c = 172690/1270 = 135.98$. The target: $\mu_0 = 137.036$. Error: $(135.98 - 137.036)/137.036 = -0.77\%$. $\square$

Remark 5.2 (Physical interpretation). The energy gap $E_t - E_c = \ln(\alpha^{-1})/\text{MU}$ means that the ratio of the top and charm quark masses is set by the same constant $\mu_0 = \alpha^{-1}$ that controls the overall mass formula normalization. This is a *self-referential* relationship: the electromagnetic coupling α encodes both the absolute mass scale (via $\mu_0 = \alpha^{-1}$) and the internal quark mass hierarchy (via the top/charm ratio).

In combination with Conjecture 4.1 ($E_c = \mu_0^{\text{sub}}$), this means the top quark energy is

$$E_t = \mu_0^{\text{sub}} + \frac{\ln \mu_0}{\text{MU}} = (\pi^2 + \pi) + \frac{\ln(4\pi^3 + \pi^2 + \pi)}{0.7933}, \quad (23)$$

which gives a fully geometric prediction for the top quark mass in terms of π alone.

Remark 5.3 (Closure condition). Conjecture 5.1 requires understanding why the top/charm energy difference equals $\ln(\mu_0)/\text{MU}$ and not some other fraction. One route: if the top quark sits at the *full* bulk self-lensing maximum $E_t^{\text{bulk}} = \pi + \ln(\mu_0)/\text{MU} + E_c - \pi = E_c + \ln(\mu_0)/\text{MU}$, then Conjecture 4.1 and Conjecture 5.1 together would fix both the charm and top energies from μ_0 alone. This would require showing that the top quark is the unique quark sitting at the *maximum* of the bulk sector density, dual to the charm quark sitting at the *minimum* of the non-bulk sector.

6 Cabibbo Angle from G_2

6.1 Setup

The Cabibbo angle θ_C is the first (12-rotation) CKM matrix element, with PDG value $|V_{us}| = \sin \theta_C = 0.22431$.

The automorphism group $G_2 = \text{Aut}(\mathbb{O})$ has dimension 14 (Addendum 37 Proposition ??).

Conjecture 6.1 (Cabibbo angle from G_2). *The Cabibbo angle is*

$$\theta_C = \frac{\pi}{\dim G_2} = \frac{\pi}{14}. \quad (24)$$

Numerically: $\sin(\pi/14) = 0.22252$. *PDG:* $|V_{us}| = 0.22431$. *Error:* $(0.22252 - 0.22431)/0.22431 = -0.80\%$.

Remark 6.2 (Geometric interpretation). The G_2 group has 14 generators; the fundamental domain of G_2 action on the imaginary octonions has angular period $2\pi/14 = \pi/7$ in the relevant circle. The Cabibbo angle $\pi/14$ is half this period, suggesting it tiles the G_2 action space with period $2\theta_C = \pi/7$.

In the $J_3(\mathbb{O})$ language of Addendum 37 Conjecture ??, the CKM matrix arises from the phase structure of the off-diagonal entries $x_{12}, x_{13} \in \mathbb{O}$ under $SU(3) \subset G_2$. The angle θ_C is the smallest non-trivial rotation in G_2 that maps one generation to another through the off-diagonal action.

Remark 6.3 (Comparison with Wolfenstein parametrization). The Wolfenstein parameter $\lambda = \sin \theta_C \approx 0.225$ organizes the CKM hierarchy as $(1, \lambda, \lambda^2, \lambda^3)$. The prediction $\lambda = \sin(\pi/14)$ would fix λ geometrically and propagate to $|V_{cb}| \approx \lambda^2$, $|V_{ub}| \approx \lambda^3$ via the Wolfenstein expansion.

These secondary predictions give $|V_{cb}| \approx 0.0495$ (PDG: 0.0410, error +20%) and $|V_{ub}| \approx 0.0110$ (PDG: 0.00369, error +200%). The higher-order elements are *not* predicted at this accuracy; the Wolfenstein expansion itself introduces approximation errors at $O(\lambda^2)$ and higher. Only θ_C (equivalently $|V_{us}|$) is claimed by Conjecture 6.1.

Remark 6.4 (Closure condition). Conjecture 6.1 requires showing that the $\mathbb{Z}_3$ -equivariant CKM phase from the $J_3(\mathbb{O})$ off-diagonal is $\pi/14$ rather than some other fraction. Specifically: the first off-diagonal mixing angle between generations 1 and 2 should be the minimal angle in the fundamental Weyl chamber of G_2 , which has size $\pi/14$ for the long root. A proof requires the full G_2 representation theory analysis of $J_3(\mathbb{O})$ deferred to Phase 4.

7 Summary and Open Problems

7.1 Results table

Result	Status	Ref	Predicted	PDG / Error
Muon sub-fold dead-end	Proven	Prop. 2.1	$m_{\text{pred}}/m_e \approx 3300$	wrong scale
Neutrino TBM from $\mathbb{Z}_3$	Consistent	Prop. 3.1	$\sin^2 \theta_{12} = 1/3$ $\sin^2 \theta_{23} = 1/2$ $\theta_{13} = 0$	0.307 (−8%) 0.545 (+9%) 0.0218 (open)
Absolute neutrino mass	Type 1 gap	Prop. 3.3	not constrained	—
Charm at μ_0^{sub}	Conjecture A	Conj. 4.1	1285 MeV	1270 MeV (+1.2%)
$m_t/m_c = \alpha^{-1}$	Conjecture B	Conj. 5.1	137.036	135.98 (−0.77%)
Cabibbo = $\pi/14$	Conjecture C	Conj. 6.1	0.22252	0.22431 (−0.80%)

7.2 Open problems

- OP1. (Phase 1.5)** Derive m_μ/m_e from the S^3/S^1 boundary fold-map. The target energy is $E_{\text{self}}^{(S^3)}(0.759) = 16.611$ (Remark 2.3). Requires extending the Paper 04 fold formalism to the sub-boundary.
- OP2. (Phase 3.5)** Derive neutrino mass-squared differences $\Delta m_{21}^2, \Delta m_{31}^2$ from the G_2 -invariant phases of the $J_3(\mathbb{O})$ off-diagonal entries $x_{ij} \in \mathbb{O}$.
- OP3. (Phase 4.1)** Prove Conjecture 4.1: charm energy = μ_0^{sub} from a balance condition on the $S^3 \cup S^1$ sub-manifold. Requires the quark color sector treatment of Addendum 37 Conjecture ??.
- OP4. (Phase 4.2)** Prove Conjecture 5.1: top/charm gap = $\ln(\mu_0)/\text{MU}$. A natural route: show that the top quark sits at the bulk sector maximum dual to the charm quark at the non-bulk minimum.
- OP5. (Phase 4.3)** Prove Conjecture 6.1: $\theta_C = \pi/\dim(G_2)$ from the G_2 Weyl chamber on $J_3(\mathbb{O})$. Requires the full CKM phase analysis of off-diagonal octonion entries.
- OP6. (Phase 4.4)** Fix θ_{13} (the reactor angle) and the CP phase δ from $J_3(\mathbb{O})$ off-diagonal phases. These are the remaining PMNS parameters beyond the $\mathbb{Z}_3$ fixed point.
- OP7. (Phase 5)** Absolute neutrino mass from the seesaw/Higgs sector. Requires extending $\hat{\mathcal{O}}$ (Paper 18) to include the Higgs coupling, currently outside the scope of $\rho(x)$.

7.3 Structural picture

The three conjectures A, B, C collectively suggest that the quark mass hierarchy is organized by exactly three geometric invariants of the $\rho(x)$ density and G_2 action:

- $\mu_0^{\text{sub}} = \pi^2 + \pi$ sets the charm mass (boundary/edge threshold).
- $\ln(\mu_0)/\text{MU}$ sets the top/charm gap (full-density self-reference).
- $\pi/14$ sets the Cabibbo angle (minimal G_2 rotation angle).

Together these fix m_c , m_t , and θ_C in terms of π alone (since $\mu_0 = 4\pi^3 + \pi^2 + \pi$ and $\dim G_2 = 14$ is an integer from the Lie algebra). The remaining quark masses (u, d, s, b) do not yet exhibit analogous relationships and are designated as the Phase 4 mass-gap problem: whether they arise from other $J_3(\mathbb{O})$ invariants or require higher-order geometry is an open question.

References

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