

Bisection Residue: Self-Similarity Obstruction and the Migration of Binary Structure

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Abstract

When a hypercube cannot be bisected into two self-similar copies, what symmetry — if any — survives? We show that the obstruction relocates the binary structure rather than eliminating it: the number 2 migrates from its role as a self-similarity count to the order of a residual symmetry group. The precise condition for $\mathbb{Z}_2$ to appear as this residue is algebraic — a *coset involution criterion* involving the stabilizer of the bisection (Theorem ??). We exhibit a compact subset of $\mathbb{R}^2$, the *chiral pinwheel*, in which the minimal residual bisection symmetry is $\mathbb{Z}_4$, not $\mathbb{Z}_2$, demonstrating that the residue depends on the geometric structure of the bisection and not merely on the obstruction.

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1 Introduction

A *rep- k tile* is a shape admissible for dissection into k mutually congruent pieces each similar to the original. The theory of rep-tiles, introduced by Golomb [?] and developed extensively since [?], includes the basic observation that a hypercube $[0, 1]^n$ is a rep- k tile only for $k = m^n$ with m a positive integer (Proposition ?? below). In particular, hypercubes are not rep-2 tiles for $n \geq 2$.

This paper investigates what replaces self-similar bisection when it is obstructed. The replacement condition is weaker: rather than requiring each piece to resemble the whole, we ask only that the two pieces resemble *each other*. This shifts the governing structure from a self-similarity equation to a group action, and raises the question: what is the minimal symmetry group witnessing the resulting bisection?

We call this the *bisection residue*. Our main results are:

- (A) A necessary and sufficient algebraic condition for the bisection residue to be $\mathbb{Z}_2$ — the *coset involution criterion* (Theorem ??).
- (B) The quaternion group Q_8 provides a minimal algebraic example where the residue is $\mathbb{Z}_4$ (Theorem ??).
- (C) Every compact subset of $\mathbb{R}^n$ admits at least one bisection with $\mathbb{Z}_2$ residue (Theorem ??).
- (D) The chiral pinwheel — a compact subset of $\mathbb{R}^2$ with 4-fold rotational symmetry and chiral arms — admits a natural bisection whose minimal residue is $\mathbb{Z}_4$, not $\mathbb{Z}_2$ (Theorem ??).

Result (D) shows that $\mathbb{Z}_2$ is not universally forced by the self-similarity obstruction: the residue depends on the *type* of bisection available, not merely on the obstruction itself.

Throughout, G denotes an isometry group acting on a metric space X , and $G_A = \{g \in G : g(A) = A\}$ denotes the stabilizer of a piece A .

2 The Self-Similarity Obstruction

We record the obstruction to self-similar bisection. These results are known; we include them to fix notation and make the paper self-contained.

Proposition 2.1 (Tiling count; cf. [?]). *If $[0, 1]^n$ is partitioned into m mutually congruent axis-aligned n -cubes, then $m = k^n$ for some positive integer k .*

Proof. Each cube has side length s ; the intervals of length s along each coordinate axis partition $[0, 1]$, forcing $s = 1/k$ for some $k \in \mathbb{Z}^+$. The total count is then k^n . $\square$

Corollary 2.1. *$[0, 1]^n$ is not a rep-2 tile for $n \geq 2$.*

Proof. $m = 2$ requires $k = 2^{1/n}$, which is not a positive integer for $n \geq 2$, since $1^n < 2 < 2^n$. $\square$

Remark 2.1. The same argument shows that 2 is not of the form k^n for any $n \geq 2$, because it is not a perfect n -th power. The following proposition reinforces this at the metric level.

Proposition 2.2 (Scale irrationality). *For $n \geq 2$, $2^{1/n}$ is irrational.*

Proof. Suppose $2^{1/n} = p/q$ in lowest terms. Then $p^n = 2q^n$, so p^n is even, hence p is even. Write $p = 2r$; then $2^{n-1}r^n = q^n$, so q^n is even, hence q is even — contradicting $\gcd(p, q) = 1$. $\square$

The two propositions attack the same root from different directions: the counting argument says 2 is not in the sequence $\{k^n\}$; the metric argument says the required scaling factor $2^{-1/n}$ cannot be rational.

3 The Residue Criterion

After ruling out self-similar bisection, we relax: seek any partition $\{A, B\}$ of X such that A and B are isometric to each other (not to the whole).

Definition 3.1 (Bisection residue). The *bisection residue* of a partition $\{A, B\}$ of X with $A \cong B$ is the cyclic group $\mathbb{Z}_m$, where m is the minimal order of any isometry witnessing $A \cong B$.

Let G act on X by isometries and let $\sigma \in G$ satisfy $\sigma(A) = B$. Since σ is bijective and $B = X \setminus A$, we have $\sigma(B) = A$, hence $\sigma^2 \in G_A$. The full set of witnessing isometries is the coset σG_A ; the bisection has $\mathbb{Z}_2$ residue if and only if σG_A contains an element of order 2.

When $X \subset \mathbb{R}^n$, two choices of G are natural. Taking $G = \text{Isom}(X)$ gives the *intrinsic residue*, which depends only on the symmetries of X itself. Taking $G = E(n)$, the full Euclidean group, gives the *ambient residue*, which allows isometries of $\mathbb{R}^n$ that need not preserve X as a set. Sections ??–?? work intrinsically; Sections ??–?? work with ambient isometries, where $E(n)$ supplies involutions independent of the internal symmetry of X . The two notions coincide when the minimal witness happens to lie in $\text{Isom}(X)$, and can differ otherwise — Theorem ?? exhibits both in the same example.

Theorem 3.1 (Coset involution criterion). *The coset σG_A contains an element of order 2 if and only if there exists $h \in G_A$ such that*

$$\sigma h \sigma = h^{-1}. \quad (1)$$

Proof. ($\Rightarrow$): Suppose $(\sigma h)^2 = e$ for some $h \in G_A$. Then $\sigma h \sigma h = e$, so $\sigma h \sigma = h^{-1}$.

($\Leftarrow$): Suppose $\sigma h \sigma = h^{-1}$ for some $h \in G_A$. Then

$$(\sigma h)^2 = \sigma h \cdot \sigma h = (\sigma h \sigma) \cdot h = h^{-1} \cdot h = e.$$

So $\sigma h \in \sigma G_A$ is an involution. $\square$

Remark 3.1. Condition (??) says that σ *inverts* h by conjugation — equivalently, h and h^{-1} are conjugate in G via σ . This is a condition on the stabilizer element h , not on σ itself. The Q_8 example (Theorem ??) illustrates why the condition can fail globally: the unique involution -1 of Q_8 is central, so it lies in every stabilizer G_A but can never be inverted by conjugation (since $j(-1)j = -j^2 = 1 \neq (-1)^{-1} = -1$), and no element $h \in H$ satisfies $jhj = h^{-1}$.

Corollary 3.1 (Trivial stabilizer forces $\mathbb{Z}_2$). *If $G_A = \{e\}$, then $\sigma^2 \in G_A$ forces $\sigma^2 = e$. The unique witnessing isometry σ is itself an involution; $\mathbb{Z}_2$ residue holds necessarily.*

Corollary 3.2 (Self-witnessing involutions). *If σ has order 2, then taking $h = e$ gives $\sigma e \sigma = \sigma^2 = e = e^{-1}$. The criterion is satisfied; $\mathbb{Z}_2$ residue holds.*

4 The Minimal Algebraic Counterexample

Without a constraint on the stabilizer, $\mathbb{Z}_2$ residue can fail. The quaternion group Q_8 provides the minimal example.

Theorem 4.1 (Q_8 bisection has no $\mathbb{Z}_2$ witness). *Let $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ act on itself by left multiplication, $H = \langle i \rangle = \{1, -1, i, -i\}$, $\sigma = j$. The partition $\{A = H, B = jH\}$ satisfies $A \cong B$ via $\sigma = j$, which has order 4. No involution witnesses $A \cong B$.*

Proof. The unique element of order 2 in Q_8 is -1 . We verify that criterion (??) fails for every $h \in H$:

h	$\sigma h \sigma = j h j$	h^{-1}	Equal?
1	$j \cdot 1 \cdot j = j^2 = -1$	1	No
-1	$j(-1)j = -j^2 = 1$	-1	No
i	$j \cdot i \cdot j = (ji)j = (-k)j = i$	$-i$	No
$-i$	$j(-i)j = -(jij) = -i$	i	No

The criterion fails for every $h \in H$, so $jH = \{j, -j, k, -k\}$ contains no involution. The minimal-order witnessing isometry has order 4; the residue is $\mathbb{Z}_4$. $\square$

Remark 4.1. The failure requires three simultaneous conditions: (i) the unique involution -1 is central, so it normalizes every subgroup; (ii) every subgroup of Q_8 contains -1 as the unique element of order 2; (iii) therefore $-1 \in A$ for every coset bisection, forcing -1 to *stabilize* A rather than swap it with B . If any condition fails — if the group has non-central involutions, or if the unique involution does not lie in every subgroup — $\mathbb{Z}_2$ residue is restored.

5 The Geometric Program

We now turn to compact subsets of $\mathbb{R}^n$, where the ambient Euclidean group $E(n)$ provides additional resources.

Theorem 5.1 (Halfspace bisections have $\mathbb{Z}_2$ residue). *Let $X \subset \mathbb{R}^n$ be compact. Let H be a hyperplane and $\{A, B\} = \{X \cap H^+, X \cap H^-\}$. The reflection τ_H across H restricts to an isometric bijection $A \rightarrow B$ and has order 2; the bisection has $\mathbb{Z}_2$ residue.*

Proof. $\tau_H : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isometry with $\tau_H^2 = \text{id}$. It maps H^+ to H^- , hence restricts to an isometric bijection $A = X \cap H^+ \rightarrow X \cap H^- = B$. $\square$

Corollary 5.1 (Existence of $\mathbb{Z}_2$ bisection). *For any compact $X \subset \mathbb{R}^n$, there exists at least one bisection with $\mathbb{Z}_2$ residue.*

Proof. Since X is compact, there exists a hyperplane H with $X \cap H^+ \neq \emptyset$ and $X \cap H^- \neq \emptyset$. Apply Theorem ?? . $\square$

Theorem ?? uses an *ambient* isometry of $\mathbb{R}^n$ — one that need not preserve X as a set. This is why it succeeds where Q_8 fails: the ambient Euclidean space supplies involutions regardless of the intrinsic symmetry of X .

6 The Chiral Pinwheel: A Geometric $\mathbb{Z}_4$ Counterexample

Theorem ?? guarantees $\mathbb{Z}_2$ for halfspace bisections. We now exhibit a compact set $X \subset \mathbb{R}^2$ admitting a *non-halfspace* bisection whose minimal residue is $\mathbb{Z}_4$.

Construction. Let $\alpha \subset \{(x, y) : x > 0, y > 0\}$ be a compact connected set satisfying:

(C1) α is *chiral*: $\text{Isom}(\alpha) = \{e\}$.

(C2) α lies strictly in the open northeast quadrant, away from the axes.

Such α exists explicitly: take α to be the filled triangle with vertices $(1, 2)$, $(3, 1)$, $(4, 4)$. Its three side lengths $\sqrt{5}$, $\sqrt{10}$, $\sqrt{13}$ are all distinct, so α is scalene and admits no reflective symmetry; combined with having no rotational symmetry, $\text{Isom}(\alpha) = \{e\}$, confirming (C1). All vertices lie in the open northeast quadrant, confirming (C2).

Let R denote counterclockwise rotation by 90° about the origin. Define

$$X = \alpha \cup R(\alpha) \cup R^2(\alpha) \cup R^3(\alpha),$$

with arms $a_0 = \alpha$, $a_1 = R(\alpha)$, $a_2 = R^2(\alpha)$, $a_3 = R^3(\alpha)$, and bisection

$$A = a_0 \cup a_2, \quad B = a_1 \cup a_3.$$

Proposition 6.1. (a) $R(A) = B$, so $A \cong B$ via R , which has order 4.

(b) $\text{Isom}(X) = \{e, R, R^2, R^3\} \cong \mathbb{Z}_4$.

(c) $G_A = \{e, R^2\}$.

Proof. (a) $R(a_0) = a_1$ and $R(a_2) = a_3$, so $R(A) = B$.

(b) Any isometry of X permutes the four arms, so $\text{Isom}(X)$ is finite. Since X is bounded, no non-trivial translation preserves X . Glide reflections have infinite order and are therefore absent from any finite isometry group. Since each arm is a rotation of α and α is chiral, no reflection preserves X . The only remaining isometries are rotations; since X has exactly 4-fold rotational symmetry, $\text{Isom}(X) \cong \mathbb{Z}_4$.

(c) $R^2(a_0) = a_2$ and $R^2(a_2) = a_0$, so $R^2(A) = A$. Neither R nor R^3 fixes A . $\square$

Theorem 6.1 (Chiral pinwheel — minimal residue is $\mathbb{Z}_4$). *For the chiral pinwheel X with bisection $\{A, B\}$ as above:*

(i) (Intrinsic.) *No element of $\text{Isom}(X)$ of order 2 witnesses $A \cong B$.*

(ii) (Ambient.) *No element of $E(2)$ of order 2 witnesses $A \cong B$.*

In particular, both the intrinsic and ambient residues of $\{A, B\}$ are $\mathbb{Z}_4$.

Proof. (i) The only involution in $\text{Isom}(X) \cong \mathbb{Z}_4$ is R^2 . By Proposition ??(c), $R^2(A) = A$, so R^2 does not witness $A \cong B$.

(ii) Elements of $E(2)$ of order 2 are reflections across lines and 180° rotations about points.

Reflections. Suppose $\rho_\ell(A) = B$. Then $\rho_\ell(a_0)$ must equal one of the arms in B as a subset of $\mathbb{R}^2$ (by disjointness and connectivity of the arms).

If $\rho_\ell(a_0) = a_1 = R(a_0)$, then $\rho_\ell \circ R^{-1}$ maps a_0 to a_0 isometrically, so $\rho_\ell \circ R^{-1} \in \text{Isom}(a_0) = \{e\}$ by (C1), giving $\rho_\ell = R$. But ρ_ℓ is orientation-reversing and R is orientation-preserving — contradiction.

If $\rho_\ell(a_0) = a_3 = R^3(a_0)$, then $\rho_\ell \circ R^{-3} \in \text{Isom}(a_0) = \{e\}$, so $\rho_\ell = R^3$. Again ρ_ℓ is orientation-reversing and R^3 is orientation-preserving — contradiction.

In both cases we reach a contradiction; no reflection maps A to B .

180° rotations. Let ρ_c denote the 180° rotation about $c \in \mathbb{R}^2$. Suppose $\rho_c(A) = B$; then $\rho_c(a_0) \in \{a_1, a_3\}$.

If $\rho_c(a_0) = a_1 = R(a_0)$, then $\rho_c \circ R^{-1} \in \text{Isom}(a_0) = \{e\}$ by (C1), so $\rho_c = R$. But R has order $4 \neq 2$, contradicting ρ_c being a 180° rotation.

If $\rho_c(a_0) = a_3 = R^3(a_0)$, then $\rho_c \circ R^{-3} \in \text{Isom}(a_0) = \{e\}$, so $\rho_c = R^3$, again order $4 \neq 2$. Contradiction.

Since no order-2 element of $E(2)$ witnesses $A \cong B$, and R (order 4) does, the minimal residue is $\mathbb{Z}_4$. $\square$

Remark 6.1. The proof of part (ii) uses condition (C1) essentially: if α had nontrivial isometries, the composition $\rho_c \circ R^{-1}$ could be a nontrivial element of $\text{Isom}(\alpha)$ rather than the identity, and the contradiction would not follow. Chirality is not merely a convenience — it is the property that forces the argument through.

7 Discussion

7.1 The three-level structure

The results naturally separate into three regimes, summarised in Table ??.

Table 1: Obstruction type and bisection residue by setting

	Setting	Obstruction type	Minimal bisection residue
$[0, 1]^n$, $n \geq 2$		Dimensional arithmetic (rep-tile)	$\mathbb{Z}_2$ (halfspace bisection)
Q_8 coset bisection		Algebraic (central involution in every subgroup)	$\mathbb{Z}_4$
Chiral pinwheel		Geometric (chirality + 4-fold rotation)	$\mathbb{Z}_4$

The common algebraic mechanism in the two $\mathbb{Z}_4$ cases is Theorem ??: the witnessing isometry σ fails to invert any element of the stabilizer G_A by conjugation.

7.2 Migration of binary structure

In the self-similar setting, the number 2 appears as the *count* of pieces. In the residue setting, 2 appears as the *order* of the minimal witnessing symmetry. The self-similarity obstruction relocates 2 between categorical roles — from cardinality to group order — rather than eliminating it. The chiral pinwheel shows that this relocation is not automatic: the value that the binary structure migrates *to* depends on the geometry of the available bisections.

7.3 Open problem

Conjecture 7.1 (Order characterisation of bisection residue). *Let $X \subset \mathbb{R}^n$ be compact and $\{A, B\}$ a bisection of X with $A \cong B$. Define*

$$k(A, B) := \min\{k \in \mathbb{Z}^+ : \exists \varphi \in E(n) \text{ with } \text{ord}(\varphi) = 2k \text{ and } \varphi(A) = B\}.$$

Then the bisection residue of $\{A, B\}$ is $\mathbb{Z}_{2k(A, B)}$.

Theorems ?? and ?? confirm this conjecture in the two cases studied: $k(A, B) = 1$ for every halfspace bisection (ambient reflection witnesses, giving $\mathbb{Z}_2$ residue), and $k(A, B) = 2$ for the chiral pinwheel bisection (R witnesses with order 4, no order-2 element does, giving $\mathbb{Z}_4$ residue). The general case — whether the order of the minimal ambient witness always determines the residue — remains open.

References

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