

Addendum 347b: The Variational / Monotone-Operator Structure

$\lambda_{\min}$ is concave, but no constant SPD metric makes the map
monotone: the obstruction is a metric-invariant ζ -soft eigenvalue
— P18-T2 sub-result (iii), Angle 3
(Addendum to Paper 18)

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Abstract

Angles 1 (345b, line test) and 2 (346b, offset–shape) both reduce the lone non-floor gap of P18-T2 sub-result (iii) to the *same* missing lever: a closed-form sign-control of the ζ -carrying Jacobian quadratic form. Angle 3 attacks that lever directly through the variational structure of the eigenvalues of $H(c) = H_0 + \sum_i c_i S_i$ ($S_i = \rho$, character I , R_{sym} , M fixed; c the scalar weights). Four findings on the genuine eigendecomposition ($V=0$ hard gate passed). **(1)** $\lambda_{\min}(c) = \inf_{\psi} \langle \psi | H(c) | \psi \rangle$ is an infimum of affine functions, hence **concave** in c (verified, 0/200 midpoint violations); higher eigenvalues are concave-minus-concave, so no global-convexity argument forces injectivity directly. **(2) The prize attempt:** a C^1 map is globally injective on a convex set if it is *monotone* in some inner product, i.e. the symmetrised Jacobian $D D\Phi + D\Phi^\top D$ is sign-definite for a single constant SPD metric D . On the true operator the Euclidean symmetrised Jacobian is *indefinite* (one persistent eigenvalue $\sim [-0.032, -0.028]$), and the best well-conditioned SPD metric ($\text{cond} \leq 10^3$) drives the box-minimum symmetrised eigenvalue only to $\sim -10^{-3}$, never positive; a degenerate metric reaches a spurious $\sim -10^{-55}$, a $\text{cond} \rightarrow \infty$ artifact, not a certificate. **(3)** The γ column of $D\Phi$ is exactly c -independent ($\sim 10^{-15}$) — the variational form of A346’s character block, the one closed-form sector. **(4)** The residual $R(c) = \|\Phi(c) - \Phi^*\|^2$ is *non-convex* (a zero-eigenvalue ζ valley; Hessian not positive-definite at 63/86 box points). **Verdict: PARTIAL** with a genuine new structural fact — the monotone-operator obstruction is a single metric-invariant ζ -soft eigenvalue — and the prize honestly not delivered. The variational route lands on the same ζ -soft boundary as Angles 1 and 2 and explains it. All claims pinned by `verify_P347b.py` (16/16).

1 The shared lever

Both prior angles converge on one residual. A345’s line test is met numerically but its clean analytic sufficient condition ($v^\top Jv$ sign-constancy) fails on $\sim 1\%$ of chords; A346’s shape block closes but the ζ -coupling does not vanish. In each case the missing piece is a closed-form statement that the Jacobian quadratic form keeps a definite sign in the ζ direction. The variational structure is the natural place to seek it: the operator is *affine* in the weights,

$$H(c) = H_0 + \sum_i c_i S_i, \quad S_i \in \{\rho, \text{character } I, R_{\text{sym}}, M\} \text{ fixed}, \quad (1)$$

so its eigenvalues inherit variational convexity properties in c . We test whether those properties deliver the lever.

2 Concavity of $\lambda_{\min}$ — clean but insufficient

Proposition 2.1 ($\lambda_{\min}$ is concave; the differences are not). *Since $H(c)$ is affine in c , $\lambda_{\min}(c) = \inf_{\|\psi\|=1} \langle \psi | H(c) | \psi \rangle$ is an infimum of functions affine in c , hence concave in c . This is verified numerically (0/200 midpoint-concavity violations). By Courant–Fischer the higher eigenvalues are min–max values, hence differences of concave functions; the matched difference-functionals $F_i = \lambda_{m_i} - \lambda_{(1,2)}$ are concave-minus-concave, with no definite curvature.*

Remark 2.2 (Why concavity does not close it). A globally concave (or convex) coordinate map need not be injective, and in any case the relevant objects are the *differences* F_i , which carry no definite curvature. So no global-convexity argument forces injectivity directly — precisely the structural reason the gap is hard. Concavity of $\lambda_{\min}$ is real and clean, but it is not the lever.

3 The monotone-operator test

Theorem 3.1 (Monotone-operator criterion). *A C^1 map Φ is globally injective on a convex set if it is monotone in some inner product: $\langle D(\Phi(c) - \Phi(c')), c - c' \rangle$ has one sign for a constant SPD metric D . By the mean-value theorem this is equivalent to sign-definiteness of the symmetrised Jacobian $DD\Phi(c) + D\Phi(c)^\top D$ for all $c = a + v^\top (DD\Phi)_{\text{sym}} v$ sign-control, exactly the closed-form lever Angles 1 and 2 need.*

We test this on the genuine eigendecomposition, with the gradient supplied in closed form by Hellmann–Feynman, $\partial\lambda_n/\partial c_i = \langle \psi_n | S_i | \psi_n \rangle$ on the true mixed eigenstate.

Proposition 3.2 (The obstruction is metric-invariant). *Euclidean ($D = I$): the symmetrised Jacobian is indefinite — one persistent negative eigenvalue (normalised coordinates $\sim [-0.032, -0.028]$) and three positive; the negative eigenvector mixes γ, ζ, β nearly equally, with the soft direction the ζ sector. Best well-conditioned metric ($\text{cond} \leq 10^3$): the box-minimum symmetrised eigenvalue is driven up to $\sim -10^{-3}$ but cannot cross zero; the map is at best semidefinite-monotone (on the boundary), never strictly monotone in any non-degenerate metric. Degenerate metric (collapsing the β weight): reaches a spurious $\sim -10^{-55}$ — a $\text{cond} \rightarrow \infty$ numerical artifact, flagged and excluded, not a certificate. The obstruction is therefore a single near-zero symmetrised eigenvalue in the ζ direction that no constant SPD metric can lift.*

Remark 3.3 (The mechanism). The offset-functional construction couples the soft ζ sector into every difference row, so the symmetrised Jacobian inherits one near-zero eigenvalue regardless of the metric. This is the same ζ -soft floor named in A330/A332/A341/A345 — now seen as a *metric-invariant* property of the monotone-operator form, which is the new structural fact of Angle 3.

4 The frozen γ column and the non-convex residual

Proposition 4.1 (The one closed-form sector). *The γ column of $D\Phi$ is exactly c -independent (maximum deviation $\sim 10^{-15}$ over the box): the γ sector is a scalar multiple of I , so $\partial\lambda_n/\partial\gamma = \text{Re}\omega^l$ is frozen. This is the variational form of A346’s exact character block — the one sector the variational structure pins in closed form. It is necessary but not sufficient: with γ frozen, the residual sign-control still requires the ζ -soft $(\alpha, \zeta, \beta) 3 \times 3$, the boundary case of Proposition 3.2.*

Proposition 4.2 (The residual is non-convex). *The residual $R(c) = \|\Phi(c) - \Phi^*\|^2$ has a Hessian with a zero eigenvalue at canonical (the ζ -flat valley of A330/A332) and is not positive-definite at*

the majority of sampled box points (63/86). So the “unique critical point from a positive-definite Hessian” route fails: the ζ valley makes R only quasi-flat along ζ , not strictly convex. (A330’s second-order strict local minimum at canonical is a statement about R restricted to the residual manifold; the full 4×4 c -Hessian is rank-deficient along ζ — consistent, not contradictory.)

Status

On the genuine eigendecomposed corpus APS operator-with-potential ($V=0$ hard gate passed, $(2n+l+2)^2$ every l -sector, not Dirichlet, not a shift), the variational / monotone-operator structure yields: (i) $\lambda_{\min}(c)$ is concave (infimum of affine), and the matched difference-functionals are concave-minus-concave, so no global-convexity argument forces injectivity directly; (ii) the chosen-functional map is not a monotone operator in the Euclidean metric (symmetrised Jacobian indefinite, one persistent ~ -0.03 eigenvalue), and no well-conditioned constant SPD metric makes it strictly monotone (best $\text{cond} \leq 10^3$ box-minimum eigenvalue $\sim -10^{-3}$, never positive; the only “monotone” metric is the degenerate $\text{cond} \rightarrow \infty$ limit, a numerical artifact, not a certificate); (iii) the γ column of $D\Phi$ is exactly c -independent — the one closed-form sector; (iv) the residual $R(c)$ is non-convex (zero-eigenvalue ζ valley, Hessian not positive-definite at 63/86 box points), so the residual-convex-program route fails. The key prize — a closed-form sign-control of the ζ -carrying Jacobian quadratic form — is **not delivered**: the variational route lands on the same ζ -soft boundary as Angles 1 and 2 and characterises it sharply (the symmetrised Jacobian is semidefinite-at-best on the ζ direction in every non-degenerate metric, because the offset-functional construction couples the soft ζ sector into every difference row). This is an honest **PARTIAL** with a genuine new structural fact (the monotone-operator obstruction is the ζ -soft eigenvalue, metric-invariant) and the prize honestly not delivered — no fourth overclaim on this thread (after A333 Dirichlet-as-corpus, A340 shift-model, A342 total-positivity dead-end). The residual across all three angles is the same named floor: a closed-form sign-definite control of the ζ -soft direction; the corpus tie $\zeta = \alpha^{5/4}$ (P18 l.348) removes ζ and remains the cleanly-closed corpus-posed-family case. A294’s re-typing of P18-T2 stands; (i)=A328/A329, (ii)=A310, (iii) A335 in-box injectivity + A341 constant-sign determinant + A345 line test + A346 shape block all unchanged; no canon .tex edited and no published number changes. Verifier: `verify_P347b.py`, 16/16 pass.

Relation to the other angles. A347’s negative result — no *constant* metric works — forces the remaining hope to a *c-dependent* certificate, which is exactly a Schur complement (the soft ζ block conditioned on the stiff shape block). That is carried out in 348b.