

Addendum 345b: The Global-Univalence Line Test

Upgrading constant-sign-determinant local injectivity to global injectivity on the admissible box — P18-T2 sub-result (iii), Angle 1

(Addendum to Paper 18)

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Abstract

The lone non-floor gap remaining across both hearts of P18-T2 (the assembly-uniqueness theorem of the master operator) is a fully rigorous proof that the free-weight coefficient-to-spectrum map $\Phi : c = (\alpha, \gamma, \zeta, \beta) \mapsto$ the four chosen spectral functionals is *globally* injective on the admissible box, evaluated on the corpus APS operator. Addendum A341 established, on a genuine eigendecomposition of the operator-with-potential $H(c) = H_0 + \alpha \rho + \zeta R_{\text{sym}} + \beta M + V_{\text{self}}$, that the 4×4 difference-functional Jacobian $J_F(c)$ has a *constant-sign*, bounded-away-from-zero determinant ($|\det| \in [4.38, 5.44]$) across the box — giving local injectivity *everywhere*. But a constant-sign nonsingular determinant gives local injectivity *only*: the map $(x, y) \mapsto (e^x \cos y, e^x \sin y)$ is a local diffeomorphism on all of $\mathbb{R}^2$ with $\det = e^{2x} > 0$ everywhere and is not globally injective. This addendum (Angle 1 of a three-angle attack, with 346b and 347b) tests three global-univalence theorems on the true operator. (1a) The Hadamard small-variation route is **refuted**: the Jacobian’s relative variation along box chords is $\sim 10^3$, not < 1 . (1b) The boundary-injectivity / degree-theory route is **valid** but does not lower the floor: the boundary map inherits the same ζ -softness (ζ moves the image ~ 0.037 while α moves it ~ 1.63 , a $\sim 45\times$ anisotropy). (1c) The convex-domain **line test** is the strong route: $F \in C^1$ on a convex domain, injective on every segment, is globally injective; on the true operator the within-segment velocity *direction* spread is tiny (median $\sim 1.5^\circ$, all chords $< 90^\circ$) even though the velocity *magnitude* swings $\sim 10^3\times$, so 0 lies outside the velocity hull with margin ~ 0.998 on every sampled chord. This supplies exactly the local-to-global bridge the constant-sign determinant lacks. The honest limit: the clean closed-form lever (directional-convexity sign-constancy of $v^\top Jv$) holds on 296/300 chords, not all, so the line-test hypothesis is met *numerically*, with the closed-form proof open at the same ζ -soft moment-ratio floor named in A341/A342. **Verdict: PARTIAL** — the correct theorem and the missing bridge, met numerically; no closed-form HIT. All claims pinned by `verify_P345b.py` (16/16).

1 The gap, the operator, and the local-only status

P18-T2 asserts that the assembly of the master operator $\hat{O}$ is unique across admissible forms. Sub-result (iii) reduces one face of that uniqueness to the injectivity of the free-weight coefficient-to-spectrum map: distinct weight tuples $c = (\alpha, \gamma, \zeta, \beta)$ must produce distinct spectra, or the assembly would be ambiguous. The map is taken on the corpus APS operator,

$$H(c) = H_0 + \alpha \rho + \zeta R_{\text{sym}} + \beta M + V_{\text{self}}, \quad (1)$$

diagonalised genuinely (Galerkin in the APS Bessel basis $R_{nl} = r^{-1} J_{l+1}(k_n r)$, $k_n = 2n + l + 2$, all satisfying the APS boundary condition; blocks built in the corpus r^3 measure and mass-orthonormalised; R_{sym} from its exact integration-by-parts block $-2G + \frac{1}{2}\psi_1\psi_1^\top$). The construction

passes the hard gate at $V = 0$: the spectrum is $(2n + l + 2)^2 = (4, 16, 36, 64)$ in every l -sector — the corpus APS eigen-identity (A329, P18 §171), *not* the Dirichlet operator of A333 and *not* the argument-shift model of A340.

The chosen functionals are A339's: with reference mode $(1, 2)$, the four spectral readouts are $\lambda_{(1,2)}, \lambda_{(0,0)}, \lambda_{(2,2)}, \lambda_{(1,1)}$, assembled by the fixed difference matrix L into $F = L\lambda$ (an offset plus three shape partners). Its Jacobian is $J_F(c) = L\partial\lambda/\partial c$, the entries computed by Hellmann–Feynman on the true (mixed) eigenstates.

Remark 1.1 (Why A341 is not enough). A341 proved $\det J_F(c)$ has constant sign and is bounded away from zero on the box. By the inverse function theorem this gives a local diffeomorphism at every point — local injectivity everywhere. It does *not* give global injectivity: the planar exponential $(x, y) \mapsto e^x(\cos y, \sin y)$ has $\det = e^{2x} > 0$ everywhere yet identifies points differing by 2π in y . The task of Angle 1 is a genuine global-univalence theorem whose hypotheses are met on the box.

2 Angle 1a (small variation) and 1b (degree theory)

1a — Hadamard small variation: refuted as a route. The cleanest sufficient condition for a local diffeomorphism to be global is that the Jacobian vary little relative to its conditioning: $\sup_t \|J(t) - J(0)\|_2 / \sigma_{\min}(J(0)) < 1$ along every chord. On the true operator this ratio is $\sim 10^3$ (maximum ~ 3300) along box chords. The criterion fails by three orders of magnitude.

Remark 2.1 (An honest negative). The constant-sign determinant is therefore *not* a consequence of J being nearly constant. The Jacobian swings enormously; the determinant is steady for a subtler reason (developed in §3). Recording the refutation prevents a future re-attempt of the small-variation lever.

1b — boundary injectivity and degree theory: valid, floor unchanged. The degree theorem is valid: a C^1 local diffeomorphism with constant-sign determinant on a compact convex set, injective on the *boundary*, is globally injective. It reduces global injectivity to injectivity on the boundary of the box. But the reduction does not give an easier sub-problem.

Proposition 2.2 (The boundary inherits the ζ -soft floor). *Over the box, varying ζ across its full admissible range moves the image by only ~ 0.037 , while varying α moves it ~ 1.63 — a $\sim 45\times$ anisotropy. The global minimum image-distance between distinct boundary points is $\sim 2 \times 10^{-3}$, about 10^3 times smaller than the image diameter (~ 2.56). Boundary injectivity therefore holds numerically, but with a margin set by the same ζ -soft floor as the interior problem. Degree theory lowers the dimension of the question, not the floor.*

3 Angle 1c: the convex-domain line test

Theorem 3.1 (Convex-domain line test). *Let $F \in C^1$ on a convex domain D . If F is injective on every line segment contained in D , then F is injective on D . A sufficient per-segment condition, in separating-hyperplane form: for each chord $t \mapsto c + tv$, $t \in [0, 1]$, there is a fixed covector u with*

$$u^\top F'(t) = u^\top J_F(c + tv) v > 0 \quad \text{for all } t \in [0, 1]. \quad (2)$$

Then $t \mapsto u^\top F(c + tv)$ is strictly increasing, so $F(c + v) \neq F(c)$; the same holds on every sub-segment, giving injectivity along the whole chord. Equivalently, 0 does not lie in the convex hull of the unit velocity vectors $\{J_F(c + tv) v / \|\cdot\| : t \in [0, 1]\}$.

The hyperplane form (2) is the mechanism the constant-sign determinant alone cannot supply, and it is met robustly on the true operator.

Proposition 3.2 (Ray separation on the true operator). *Along every sampled box chord, the within-segment velocity direction spread is tiny (median $\sim 1.5^\circ$, maximum $< 10^\circ$, every chord $< 90^\circ$), even though the velocity magnitude swings by $\sim 10^3$. Consequently the image is a near-monotone curve along every ray, and 0 sits outside the convex hull of the unit velocities with hull-distance margin ~ 0.998 on every chord. This is the genuine local-to-global mechanism: not small Jacobian variation (refuted, §2) but near-fixed velocity direction.*

Remark 3.3 (Reconciliation with 1a). Proposition 3.2 and Remark 2.1 are consistent: the velocity magnitude $\|J_F v\|$ swings $\sim 10^3 \times$ (so J is far from constant, 1a fails) while the velocity direction is nearly fixed (so the hyperplane test passes). It is the direction, not the magnitude, that controls injectivity along a ray.

4 The analytic handle and its honest limit

The cleanest *rigorous* sufficient condition specialising Theorem 3.1 is directional convexity: $v^\top J_F(c + tv) v$ keeps a constant sign along the chord (then $v^\top (F(c + v) - F(c)) = \int_0^1 v^\top J_F v dt \neq 0$). This is the closed-form lever; if it held on every chord, the line test would close in closed form.

Remark 4.1 (The $\sim 1\%$ violation: why the lever is numerical). On the true operator the sign-constancy $v^\top J_F(c + tv) v$ holds on $\sim 296/300$ sampled chords — not all. A $\sim 1\%$ violation rate forbids asserting it as a theorem hypothesis. The robust statement that *does* hold on every chord is the half-space / direction-spread form of Proposition 3.2, which is established numerically. Moreover $v^\top J_F v$ and $\det J_F$ are the *same* object: eigenvector-weighted Galerkin moment ratios, with no elementary closed form — the identical ζ -soft floor named in A341 and A342. The line test thus relocates the gap precisely, but does not dissolve it.

Status

On the genuine eigendecomposed corpus APS operator-with-potential ($V=0$ hard gate passed, $(2n+l+2)^2$ every l -sector, not Dirichlet, not a shift), Angle 1 upgrades the status of sub-result (iii) from “constant-sign $\det \Rightarrow$ local injectivity only” (A341) to “constant-sign $\det +$ ray separation (0 outside the velocity hull on every chord, direction spread $< 90^\circ$) $\Rightarrow$ the convex-domain line test is satisfied $\Rightarrow$ global injectivity on the box.” The small-variation route (1a) is refuted; the degree-theory route (1b) is valid but does not lower the ζ -soft floor. The line-test hypothesis is met *numerically* with a large margin; the clean closed-form lever ($v^\top J_F v$ sign-constancy) fails on $\sim 1\%$ of chords, so a fully rigorous closed-form proof remains **open** at the same ζ -soft moment-ratio floor as the determinant. This is an honest **PARTIAL**: the correct global-univalence theorem and the missing local-to-global bridge, met numerically — not a closed-form HIT. The corpus tie $\zeta = \alpha^{5/4}$ (P18 l.348) removes ζ and remains the cleanly-closed corpus-posed-family case. No canon .tex of P18 is edited and no published number changes; A335/A336/A341 status is unchanged. Verifier: `verify_P345b.py`, 16/16 pass.

Relation to the other angles. Angle 2 (346b) peels ζ off as a spectral offset and closes the residual (α, β) shape block as a P-matrix; Angle 3 (347b) shows no constant SPD metric makes the map monotone; 348b realises the residual as a single Schur-complement scalar. All four converge on the same named floor: a closed-form sign-control of the ζ -soft direction.