

Addendum 342: The Determinant-Sign Question for the ζ -Sector

Total positivity fails on the genuine **eigh**-diagonalized corpus APS operator; a pure-mode / mixing split controls A341's constant-sign determinant down to one numerical perturbation inequality, giving local injectivity everywhere — a PARTIAL

TOE Corpus

2026-06-15

Abstract

Outcome: PARTIAL (the briefed total-positivity route fails; a pure-mode / mixing split controls the sign, modulo one numerical bound).

Addresses: P18-T2 sub-result (iii), the analytic attempt to prove $\det J_F(c)$ has constant sign over the admissible box — A341's pre-registered conditional follow-up, via total positivity / generalized-Vandermonde theory (Karlin; Gantmacher–Krein), on the *genuine eigh*-diagonalized corpus APS operator-with-potential.

Verifier: `verify_P342.py` (18 PASS / 0 FAIL, exit 0).

A341 established, on a genuine **eigh**-diagonalization of $H(c) = H_0 + \alpha\hat{\rho} + \zeta R_{\text{sym}} + \beta\hat{M} + V_{\text{self}}$, that the 4×4 difference-functional Jacobian $J_F(c)$ is *not* a P-matrix but has a *constant-sign*, bounded-away-from-zero determinant ($|\det| \in [4.38, 5.44]$) over the box — a *numerical* fact giving local injectivity everywhere. A341's pre-registered conditional follow-up: make that constant-sign claim *analytic*. The natural tool is total positivity: a determinant $\det[f_i(k_j)]$ of a totally-positive kernel evaluated at *ordered* nodes has a definite sign (Gantmacher–Krein, Karlin). **The finding is a PARTIAL.** The briefed primary attack *fails*: the kernel is *not* totally positive — (a) the character row $\text{Re}(\omega^l)$ is not a function of the radial node k (it depends on $l \bmod 3$), so the generalized-Vandermonde hypothesis does not even apply to it; (b) the genuine APS moment columns $\langle \hat{\rho} \rangle, \langle r \rangle$ are *non-monotone* in k (A336's shift-model monotonicity does not survive the true mixing), so the kernel's 2×2 minors are sign-indefinite (20 positive, 16 negative). What *does* work is a pure-mode / mixing split: $\det J_F^{\text{pure}}$ (diagonals on the unmixed APS modes, no **eigh**) is basis-converged (-3.81210 at $\text{NM} = 8/10/14$) and negative, matching A341's true sign; the character column is *exactly* c -independent; the mixing perturbation is column-localized; and the first-order multilinear perturbation bound (≤ 1.728 over the box) stays below $|\det J_F^{\text{pure}}| = 3.812$, so the sign cannot flip — a constant-sign nonvanishing determinant, hence *local* injectivity at every box point. **Ceiling (explicit):** this is local injectivity everywhere — an upgrade over A341's purely numerical determinant — *not* global in-box injectivity ($\det \neq 0$ is necessary, not sufficient; cf. $(e^x \cos y, e^x \sin y)$); global in-box still rests on A335's no-second-preimage scan. The pure-mode sign is semi-analytic; the mixing bound is basis-converged and structurally explained but *numerical*, not closed-form. Verdict: PARTIAL — the total-positivity route is closed off as a dead end, and the A341 floor is *sharpened* from “a constant-sign determinant with no analytic handle” to “a constant-sign determinant reduced to a single explicit perturbation inequality, modulo one numerical bound.” Honesty over closure: a PARTIAL is reported because the TP route fails and the surviving bound is numerical — no fourth overclaim on this thread.

1 Why this probe exists

A341 redid A340’s free-weight (iii) probe on a *genuine* **eigh**-diagonalization of the corpus APS operator-with-potential and reported an honest FLOOR: A340’s P-matrix / Gale–Nikaido / β -binding mechanism was a shift-model artifact (the Jacobian is not a P-matrix on the true mixed eigenstates), but in-box injectivity *survived* by a different mechanism — the determinant of the 4×4 difference-functional Jacobian $J_F(c)$ has *constant sign* (negative) and is bounded away from zero ($|\det| \in [4.38, 5.44]$) across the box, so the Jacobian is nonsingular with fixed orientation everywhere, hence locally injective at every box point. That determinant check was *numerical*. A341’s single pre-registered conditional follow-up named the present probe: try to make “ $\det J_F(c) \neq 0$ (constant sign) for all c in the box” *analytic*.

The natural analytic tool is total positivity. Classical theory (Karlin, *Total Positivity*; Gantmacher–Krein) says that a determinant $\det[f_i(k_j)]$ of a totally-positive (sign-regular) kernel evaluated at *ordered* nodes $k_1 < \dots < k_m$ has a *definite* sign — a generalized Vandermonde. If the columns of J_F were moment-type functions of an ordered radial mode k , and that kernel were totally positive, the constant sign would be *forced* closed-form. This addendum runs that attack on the genuine operator.

The honest ceiling, stated up front. Proving $\det J_F(c) \neq 0$ everywhere gives *local* injectivity everywhere — an upgrade over A341’s numerical determinant — *not* global in-box injectivity. A nonvanishing Jacobian is necessary but not sufficient for global univalence: the standard counterexample $(x, y) \mapsto (e^x \cos y, e^x \sin y)$ has nowhere-vanishing Jacobian yet is not injective. So the *maximum* this probe can deliver is “local injectivity everywhere, rigorously”; the global in-box no-second-preimage statement still rests on A335’s scan. We do *not* claim (iii) closes globally from a determinant-sign result.

2 The genuine operator (re-grounded from A341)

We carry A341’s construction verbatim. The corpus APS eigenstates $R_{nl}(r) = r^{-1} J_{l+1}(k_n r)$, $k_n = 2n + l + 2$, all satisfy the linear APS boundary condition (A334), so any finite combination does; we build the Galerkin matrix of the full $H(c)$ in this basis (corpus r^3 measure), mass-orthonormalize via a Cholesky factor of the Gram matrix, keep $H_0 = \text{diag}(k_n^2)$ as the corpus APS operator’s exact action (A329 / P18 1.171), assemble the genuine symmetric multiplication blocks ($\hat{\rho}$, $\hat{M} = \langle r \rangle$, V_{self}) and the exact-IBP R_{sym} block $-2G + \frac{1}{2}\psi_1\psi_1^\top$, and **eigh** it. The verifier confirms the $V = 0$ hard gate (spectrum $(2n + l + 2)^2$ in every l -sector, distinct from the Dirichlet squared-Bessel zeros), a live off-diagonal coupling (max off-diag 1.13), mixed eigenvectors, and A341’s constant-sign determinant over the box. The difference functionals are A339’s: offset $F_0 = \lambda_{(1,2)}$ and three shape differences against partners $(0, 0)$, $(2, 2)$, $(1, 1)$.

3 The briefed primary attack: total positivity — it fails

For the generalized-Vandermonde sign theorem the kernel $K = [g_i(k_j)]$ must be a function of the ordered node k and totally positive. Two obstructions, both fatal.

(a) The character row is not a function of k . The character functional returns $\text{Re}(\omega^l) \in \{1, -\frac{1}{2}, -\frac{1}{2}\}$ by l -sector ($l \bmod 3$), *not* by the radial mode k . So K is not even of the form $[f_i(k_j)]$ in its character row, and the Karlin / Gantmacher–Krein hypothesis does not apply to it at all.

(b) The moment columns are non-monotone in k . On the genuine APS moments, over the k -ordered modes $k \in \{2, 5, 7, 8\}$,

$$\langle \hat{\rho} \rangle = (269.16, 235.98, 218.74, 229.81), \quad \langle r \rangle = (0.754, 0.689, 0.650, 0.678),$$

both *non-monotone* (each rises at the last node). A336's shift-model monotonicity of $\langle r \rangle$ in k does *not* survive the genuine mode structure. A totally-positive kernel needs monotone (more precisely, sign-regular) generators; these are not. Consequently the kernel's 2×2 minors are sign-indefinite (20 positive, 16 negative): K is *not* totally positive, and the classical generalized-Vandermonde sign theorem gives *no* closed-form pure-mode sign. The briefed primary attack fails — recorded honestly.

4 What does work: a pure-mode / mixing split

Even though K is not totally positive, the determinant sign is controlled, by a different route.

The pure-mode determinant is basis-converged and negative. Reading the sector diagonals on the *unmixed* APS basis modes e_n of H_0 (no **eigh**) gives a pure-mode Jacobian J_F^{pure} with $\det J_F^{\text{pure}} = -3.81210$, stable to five digits across $\text{NM} = 8/10/14$ — a genuine continuum value (a semi-analytic quantity: a combination of Bessel-weighted moment integrals, no diagonalization needed). Its sign matches A341's true determinant sign (negative): the unmixed structure carries the sign, and the mixing is a perturbation, not a sign-reverser.

The character column is exactly c -independent. The mixing perturbation $\Delta(c) = J_F(c) - J_F^{\text{pure}}$ leaves the character column *exactly* unchanged ($\max |\Delta_{\text{char}}| = 0$ over the box). Only the moment columns move; among those, $\|\Delta_{\text{rsym}}\| \leq 0.52$, $\|\Delta_{\text{mom}}\| \leq 0.07$, and $\|\Delta_{\rho}\|$ is large in norm but acts on a column of magnitude $O(220\text{--}270)$, a small *relative* move (eigenvector off-axis component ≤ 0.157).

The first-order multilinear bound stays below $|\det J_F^{\text{pure}}|$. By multilinearity of the determinant in columns, $\det(J_F^{\text{pure}} + \Delta) = \det J_F^{\text{pure}} + (\text{column-replacement terms})$. The first-order term, summed in absolute value $\sum_j |\det(J_F^{\text{pure}} \text{ with column } j \rightarrow \Delta_j(c))|$, is ≤ 1.728 over the box, *below* $|\det J_F^{\text{pure}}| = 3.812$. Equivalently $\max_c |\det J_F(c) - \det J_F^{\text{pure}}| = 1.630 < 3.812$. So $\det J_F(c)$ keeps the pure-mode negative sign and is nonzero everywhere: a constant-sign, nonvanishing determinant, hence local injectivity at every box point.

5 Verdict

Proposition 5.1 (PARTIAL). *On the genuine **eigh**-diagonalized corpus APS operator-with-potential ($V=0$ gate passed; A341's constant-sign determinant re-confirmed), the total-positivity / generalized-Vandermonde route fails: the difference-functional kernel is not totally positive — its character row is not a function of the radial node k , and the genuine APS moment columns are non-monotone in k (so the 2×2 minors are sign-indefinite, 20 positive / 16 negative). A pure-mode / mixing split instead controls the sign: $\det J_F^{\text{pure}}$ is basis-converged and negative, the character column is exactly c -independent, and the first-order multilinear perturbation bound (≤ 1.728 over the box) stays below $|\det J_F^{\text{pure}}| = 3.812$, so the determinant is constant-sign and nonvanishing — giving local injectivity at every box point. The pure-mode sign is semi-analytic; the perturbation bound is basis-converged and structurally explained but numerical, not closed-form.*

So sub-result (iii) advances as follows: A341’s constant-sign determinant (local injectivity everywhere) is now *analytically structured* — the total-positivity route is closed off as a dead end (the kernel is provably not totally positive), and the surviving constant-sign claim is reduced to one explicit perturbation inequality (first-order multilinear bound $< |\det J_F^{\text{pure}}|$ over the box). That inequality is currently certified *numerically* (basis-converged grid), not in closed form. The ceiling stands: this is *local* injectivity everywhere, not global; the global in-box no-second-preimage still rests on A335’s scan. The corpus tie $\zeta = \alpha^{5/4}$ (P18 l.348) remains the cleanly-closed corpus-posed-family case.

Pre-registered follow-up (one, conditional). Bound the first-order multilinear perturbation term in *closed form* — e.g. via a Cauchy–Schwarz estimate of the eigenvector off-axis component times the column moment spreads, using the bounded mixing (≤ 0.157) and the frozen character column. A closed-form bound below $|\det J_F^{\text{pure}}|$ would upgrade PARTIAL to the rigorous local-injectivity proof; if the perturbation term resists a closed form, settle the mixing bound to a named floor parallel to G1 / the APS realization.

Invariants. A294’s re-typing of P18-T2 stands; (i)=A328/A329, (ii)=A310, (iii) A335 in-box injectivity unchanged; A336 per-axis monotonicity unchanged; A341’s constant-sign determinant unchanged; no canon .tex edited; no published number changes.