

Addendum 336: The Hellmann–Feynman Monotonicity on the APS Operator

The three shape sectors stay sign-definite under the corpus boundary condition, but the renormalisation sector’s diagonal sign-flips — per-axis monotonicity in ζ fails, and A333’s four-sector picture does not transfer

TOE Corpus

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Skill: gap-closure v1.1 — **Outcome:** PARTIAL + REFUTATION (three-sector monotonicity transfers to the corpus operator; ζ -monotonicity fails as the APS zeta diagonal is sign-indefinite/sign-flipping; A333’s four-sector-monotonicity HIT is a Dirichlet artifact; the analytic ζ gap is real under the corpus BC) — **Addresses:** P18-T2 (iii) APS monotonicity; A335; A334; A333; A332; A331; A330; A329; A302; A294 — **Verifier:** verify_P336.py — 17 PASS / 0 FAIL, exit 0

Abstract

A332 ran the structural Hellmann–Feynman monotonicity argument for sub-result (iii)’s free-weight coefficient injectivity. By Hellmann–Feynman $\partial\lambda_n/\partial c_i = \langle\psi_n|S_i|\psi_n\rangle$, the diagonal matrix element of sector S_i in the eigenbasis; a sector contributes monotonically to every eigenvalue iff its diagonal has a fixed sign. A332 proved three of four sectors sign-definite — $\alpha\rho$ ($\rho \geq 0$ multiplication, diagonal $116\dots181 > 0$), βM (M positive, diagonal in $[0, 1] \geq 0$), $\gamma \text{Re}\omega^k$ (a rigid nonzero shift, monotone per $\mathbb{Z}_3$ character block) — hence injective in (α, γ, β) , and found the fourth, ζR with $R = r\partial_r$, near-constant ($\langle\psi|R|\psi\rangle$ mean -2.000 , relative spread 1.7×10^{-4} , cosine with the uniform vector 1.00000): a near-uniform shift, the soft sector. A333 then claimed ζ is an exact rigid shift and completed four-sector monotonicity. But A332 and A333 built their radial operator on the interior mesh $r = \text{linspace}(dr, 1 - dr, N - 1)$ — a **Dirichlet wall** at $r = 1$ ($\psi(1) = 0$), not the corpus APS boundary condition A329 established (P18 lines 393–396). A334 built the corpus APS operator and showed the zeta diagonal there is *not* the near-uniform -2 : it is state-dependent and even sign-flipping. **This addendum redoes A332’s monotonicity on the corpus APS eigenstates.** The three shape sectors *stay* sign-definite — $\rho \geq 0$ and M positive are state-independent facts and the $\text{Re}\omega^k$ shift is rigid, so their per-axis monotonicity transfers to APS untouched ($\langle\psi|\rho|\psi\rangle > 0$ on the APS states, min 106; $\langle\psi|M|\psi\rangle$ in $[0, 1]$; $(1, -\frac{1}{2}, -\frac{1}{2})$). **The zeta sector fails.** On the APS eigenstates $\langle\psi_n|R_{\text{sym}}|\psi_n\rangle = -2 + \frac{1}{2}|\psi(1)|^2/\langle\psi|\psi\rangle$ ranges from -1.982 to $+0.238$ across modes, and crucially it sign-flips: eleven of twelve modes *lower* the eigenvalue as ζ rises, but the $l = 2, n = 0$ mode *raises* it ($+0.238 > 0$). So $\partial\lambda_n/\partial\zeta$ changes sign across modes — per-axis monotonicity in ζ *fails* on the corpus operator. A333’s “four-sector monotonicity complete” is a Dirichlet artifact (it required the near-uniform -2 , cosine 1.0 ; the APS diagonal has cosine 0.864) and does not transfer. **What survives:** per-axis injectivity for (α, γ, β) by structural monotonicity; ζ injectivity is *not* provable by monotonicity, so the analytic global proof at the zeta sector is genuinely open under the corpus BC. Jointly with A335 (the sibling), which found in-box global injectivity *survives* on the corpus operator with a *larger* moat (6.20×10^{-3} vs A331’s 1.67×10^{-3} , the state-dependent zeta diagonal stiffening the landscape): (α, γ, β) are analytically injective and ζ injectivity rests on A335’s in-box numerics, with the analytic global proof at the zeta sector open under the corpus boundary condition and the corpus tie $\zeta = \alpha^{5/4}$ the clean closure

for the corpus-posed family. The same mode-structure is read out a second way: the α - ζ spectral collinearity — A330’s 0.9986 on the Dirichlet model — drops to 0.662 on the corpus operator, breaking the soft α - ζ ray A302’s fragility rode on, which is the structural reason A335’s moat grows. Verifier `verify_P336.py`: 17 PASS / 0 FAIL. No published number changes.

1 The probe and the discipline

This is the second of A334’s two pre-registered fan-out probes, A335 (the redone bounded-box second-preimage scan) the sibling. A334 established that the whole sub-result (iii) thread used a Dirichlet model and that, under the corpus APS boundary condition, the renormalisation sector’s spectral response is genuinely soft and state-dependent. Two injectivity results inherited from the Dirichlet thread therefore needed re-establishing on the corpus operator: A331’s second-preimage scan (A335) and A332’s structural monotonicity (this probe). The binding solution-bar discipline governs — do not force closure, honesty over closure. Here that discipline is the substance: A332’s clean “three monotone + one near-uniform” picture, and A333’s upgrade of ζ to a fourth monotone sector, both lean on a property of the Dirichlet eigenstates ($\psi(1) = 0$) that the corpus eigenstates do not have. The honest discharge is to recompute the Hellmann–Feynman diagonals on the corpus operator and report what each sector does on *it*, even where the answer overturns A333.

The operator is P18’s bulk Dirac² radial form on the unit ball, $-d_r^2 - (3/r)d_r + l(l+2)/r^2$ on the r^3 measure, with regularity at $r = 0$ and APS spectral projection at $r = 1$ (A329, P18 lines 393–396). The Hellmann–Feynman diagonal is taken in the corpus r^3 inner product, in which the operator is self-adjoint.

2 The corpus APS operator (the hard gate)

The APS radial eigenfunctions are $R_{nl}(r) = r^{-1}J_{l+1}(kr)$ with $k = 2n + l + 2$ (the regular root; the singular root $r^{-1}Y_{l+1}$ is not in $L^2(r^3 dr)$, A329 check 9), reproducing the integer ladder $E_{nl} = (2n + l + 2)^2$ (P18 line 171) and *not* vanishing at the wall, $R_{nl}(1) = J_{l+1}(2n + l + 2) \neq 0$. The verifier’s hard gate (check 3) is that the no-potential spectrum comes out the integer ladder $l = 0 : (4, 16, 36, 64)$, distinct from the Dirichlet squared-Bessel zeros (14.68, 49.22, ...); if it had come out squared-Bessel zeros the probe would be back on the Dirichlet model and void. The gate passes, and the eigenstates’ boundary values are nonzero (check 4, $l = 0 : (0.577, -0.066, -0.277, 0.235)$). The Bessel implementation (numpy-only series, the same as A334/A329) is cross-checked against known J_1 zeros and a finite-difference derivative (checks 1–2).

3 The three shape sectors stay sign-definite

Proposition 3.1 (Three sectors transfer to the corpus operator). *On the corpus APS eigenstates the three shape sectors keep the sign-definite Hellmann–Feynman diagonals that make their single-coefficient maps strictly monotone, hence injective:*

- $\alpha\rho$: $\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x \geq 0$ on $[0, 1]$, a nonnegative-function multiplication, so $\langle \psi | \rho | \psi \rangle > 0$ on every APS mode and l -sector (min 106.4). Sign-definite $\Rightarrow$ monotone $\Rightarrow$ injective in α .
- βM : $M = \sum_n (\mu_n / \mu_0) P_n$ with $\mu_n \geq 0$ ($\rho \geq 0$) and P_n orthogonal projections is a positive operator, so $\langle \psi | M | \psi \rangle \in [0, 1] \geq 0$ on the APS $l = 0$ states (min 0.740). Sign-definite $\Rightarrow$ monotone $\Rightarrow$ injective in β .

- $\gamma \operatorname{Re} \omega^k$: $\partial \lambda / \partial \gamma = \operatorname{Re} \omega^k = (1, -\frac{1}{2}, -\frac{1}{2})$ is state-independent and boundary-condition-independent (it is the $\mathbb{Z}_3$ character value, carrying no eigenstate content), a rigid nonzero shift, monotone per character block.

The transfer is structural, not incidental: $\rho \geq 0$ and M positive are facts about the operators, independent of which states one evaluates them on, and the $\operatorname{Re} \omega^k$ shift is a number, not an expectation. So the part of A332’s argument that proved injectivity in (α, γ, β) is BC-independent and carries to the corpus operator untouched (checks 5–8, 12).

4 The zeta sector fails: the diagonal sign-flips

Proposition 4.1 (ζ -monotonicity fails under the corpus boundary condition). *On the corpus APS eigenstates the renormalisation diagonal*

$$\frac{\partial \lambda_n}{\partial \zeta} = \langle \psi_n | R_{\text{sym}} | \psi_n \rangle = -2 + \frac{1}{2} \frac{|\psi(1)|^2}{\langle \psi | \psi \rangle}$$

is state-dependent and ranges from -1.982 to $+0.238$ across modes (for $l = 0$: $-0.689, -1.971, -1.323, -1.261$) — not the near-uniform -2 of the Dirichlet model. Crucially it changes sign: of the twelve modes evaluated, eleven are negative (the eigenvalue lowers as ζ rises) but the $l = 2, n = 0$ mode is positive ($+0.238$, the eigenvalue rises). So some eigenvalues rise and others fall as ζ increases. The diagonal is therefore not sign-definite ($\max 0.238 > 0$, $\min -1.982 < 0$) and mode-structured (cosine with the uniform vector 0.864 , not 1.0000). Per-axis monotonicity in ζ fails.

This is the measurement A332 could not have made on the Dirichlet model, where the boundary term $\frac{1}{2}|\psi(1)|^2$ vanishes by construction and the diagonal collapses to the rigid bulk -2 . Under the corpus boundary condition the boundary term is the live, state-dependent piece A334 measured, and it is large enough at the $l = 2, n = 0$ mode ($k = 4$, $\psi(1) = J_3(4) \approx 0.43$) to flip the bulk -2 to a positive value. A sign-flipping Hellmann–Feynman diagonal is exactly the failure of the fixed-sign condition monotonicity needs: Weyl monotonicity gives a *uniform* spectral direction only when the diagonal keeps one sign, and here it does not (checks 9–11).

The α – ζ collinearity drop

The same state-structure that breaks ζ -monotonicity is visible a second way, and it is the quantity that ties this probe to its sibling A335. A330 measured the spectral collinearity of the renormalisation sector with the density sector — the cosine between the α response column $\langle \psi_n | \rho | \psi_n \rangle$ and the ζ response column $\langle \psi_n | R_{\text{sym}} | \psi_n \rangle$ across modes — as 0.9986 on the Dirichlet model: a near-perfect degeneracy, the soft α – ζ ray along which A302’s uniqueness was fragile. Recomputing the *same* cosine on the corpus APS eigenstates gives 0.662 (check 12), a drop of 0.336 . The reason is structural: on the Dirichlet model the ζ diagonal was the near-uniform -2 (cosine with the uniform vector 1.0000), and a near-uniform column is necessarily near-collinear with the always-positive α column; under the corpus boundary condition the ζ diagonal carries genuine mode-structure (cosine with uniform 0.864), so it is no longer near-degenerate with α . The collinearity drop and the cosine-with-uniform drop are two readouts of one fact (check 13). This is precisely A335’s mechanism: breaking the α – ζ degeneracy is what makes the off-canonical residuals rise faster and the in-box moat *larger* (6.20×10^{-3} vs A331’s 1.67×10^{-3}). The state-dependence that makes ζ analytically soft is the very thing that makes the numerical landscape stiffer.

5 The verdict and what it does to (iii)

Remark 5.1 (A333’s four-sector picture does not transfer; the analytic ζ gap is real). *A333’s claim that ζ is an exact rigid shift, completing four-sector monotonicity, required the Dirichlet near-uniform -2 (cosine 1.0). The APS diagonal is not that constant — it is sign-indefinite and sign-flipping — so A333’s four-sector-monotonicity HIT is a Dirichlet artifact and does not transfer to the corpus operator (check 16). The clean “three monotone + one exact-shift” picture is gone. What survives is the part that was BC-independent all along: per-axis injectivity in (α, γ, β) . ζ injectivity is not provable by monotonicity under the corpus boundary condition, so the analytic global proof at the zeta sector is genuinely open, exactly the A332 obstruction A333 wrongly believed it had removed.*

The honest joint picture closes with A335, the sibling probe (check 17). A335 redid A331’s bounded-box second-preimage scan on the same corpus APS operator and found that in-box global injectivity *survives*, with a *larger* moat (6.20×10^{-3} at $M = 6$, versus A331’s Dirichlet 1.67×10^{-3} , a $\sim 3.7\times$ ratio). The mechanism is the same state-dependence that breaks ζ -monotonicity here: the mode-structured zeta diagonal (cosine 0.864, not 1.0) breaks the near-perfect α – ζ degeneracy that made the Dirichlet valley flat, so off-canonical residuals rise faster and the in-box landscape is *stiffer*. The two probes are not in tension: injectivity holds *numerically* (A335) even though ζ -monotonicity fails *analytically* (here). So the corpus-faithful status of sub-result (iii)’s free-weight injectivity is: (α, γ, β) analytically injective by structural monotonicity; ζ injectivity resting on A335’s in-box numerical scan; the analytic global proof at the zeta sector open under the corpus boundary condition; and the corpus tie $\zeta = \alpha^{5/4}$ (P18 line 348) the clean closure for the corpus-posed family.

6 Gate ledger

Object	Gate	Finding
Corpus APS operator (hard gate)	PASSED	$(2n + l + 2)^2$ spectrum, $\psi(1) \neq 0$, not squared-B
$\alpha\rho$ on APS states	MONOTONE	$\langle \psi \rho \psi \rangle > 0$ (min 106); transfers from A332
βM on APS states	MONOTONE	M positive, $\langle \psi M \psi \rangle \in [0, 1]$; transfers
$\gamma \text{Re } \omega^k$	MONOTONE	rigid state/BC-independent shift $(1, -\frac{1}{2}, -\frac{1}{2})$
ζR on APS states	NON-MONOTONE	diagonal $-1.98 \cdots + 0.24$, sign-flips at $l = 2, n =$
α – ζ collinearity	DROPS	0.9986 (Dirichlet, A330) $\rightarrow$ 0.662 (APS); breaks
A333 four-sector monotonicity	DIRICHLET ARTIFACT	required near-uniform -2 ; does not transfer
(iii) analytic ζ injectivity	OPEN (corpus BC)	not provable by monotonicity; numerical via A3
(iii) in-box injectivity	SURVIVES (A335)	larger moat; tie the clean closure

7 Conclusion

The probe redoes A332’s Hellmann–Feynman monotonicity argument on the corpus APS operator that A334 built (the Bessel construction $R_{nl} = r^{-1} J_{l+1}((2n + l + 2)r)$, spectrum $(2n + l + 2)^2$, eigenstates that do not vanish at the wall), guarded by the hard gate that the no-potential spectrum is the integer ladder and not the Dirichlet squared-Bessel zeros. The three shape sectors transfer: $\alpha\rho$ ($\rho \geq 0$ multiplication, diagonal strictly positive on the APS states), βM (M positive, diagonal in $[0, 1]$), and $\gamma \text{Re } \omega^k$ (a rigid state- and BC-independent shift) stay sign-definite, so per-axis

injectivity in (α, γ, β) is BC-independent and carries to the corpus operator untouched. The zeta sector fails: $\langle \psi_n | R_{\text{sym}} | \psi_n \rangle$ on the APS eigenstates is state-dependent, ranges -1.982 to $+0.238$, and sign-flips (eleven modes lower, the $l = 2, n = 0$ mode raises), so $\partial \lambda_n / \partial \zeta$ changes sign and per-axis monotonicity in ζ does not hold under the corpus boundary condition. A333's four-sector-monotonicity HIT is a Dirichlet artifact and does not transfer; the analytic global proof at the zeta sector is genuinely open under the corpus operator, closed only numerically and in-box by A335 (which found injectivity survives with a larger moat, the state-dependent zeta diagonal stiffening the landscape). The honest joint picture: (α, γ, β) analytically injective, ζ injectivity resting on A335's in-box numerics, the analytic ζ gap open under the corpus boundary condition, and the corpus tie $\zeta = \alpha^{5/4}$ the clean closure for the corpus-posed family. A294's re-typing of the whole P18-T2 stands; sub-results (i) (A328/A329) and (ii) (A310) are unchanged. No canon source files are edited; no published number changes; the $\hat{O}$ assembly and its derived observables are untouched. Verifier `verify_P336.py` reaches 17 PASS / 0 FAIL and exits zero.