

## Addendum 334: The APS Boundary-Condition Reconciliation

The whole sub-result (iii) thread used a Dirichlet model; under the corpus APS boundary condition the renormalisation sector is genuinely soft, and A333's exact rigid shift is a Dirichlet artifact

TOE Corpus

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**Skill:** gap-closure v1.1 — **Outcome:** GROUNDED + REFUTATION (the corpus APS operator built and validated; the zeta sector is genuinely soft and state-dependent under the corpus BC; A333's exact-rigid-shift HIT is a Dirichlet artifact that does *not* transfer) — **Addresses:** P18-T2 (iii) APS reconciliation; A333; A332; A331; A330; A329; A302; A294 — **Verifier:** `verify_P334.py` — 16 PASS / 0 FAIL, exit 0

### Abstract

P18-T2 sub-result (iii) asks whether  $(\alpha, \gamma, \zeta, \beta)$  are the unique spectrum-matching assignment on P18's master operator. A329 established that the corpus boundary condition is APS spectral projection at  $r = 1$  (P18 lines 393–396), whose no-potential spectrum is the integer ladder  $E_{nl} = (2n + l + 2)^2$  (P18 line 171). But every numerical addendum in the (iii) thread — A302, A330, A331, A332, and crucially A333 — built its radial operator on the interior mesh  $r = \text{linspace}(dr, 1 - dr, N - 1)$ , which forces  $\psi = 0$  at *both* endpoints: a **Dirichlet wall** at  $r = 1$ . The Dirichlet  $l = 0$  spectrum is the squared Bessel zeros  $j_{1,n}^2 = (14.68, 49.22, \dots)$ , which *disagree* with the corpus APS ladder  $(4, 16, 36, \dots)$  at every mode (A329). So the (iii) thread is a Dirichlet *model*, not the corpus operator. A333 rested its central claim on this: it proved the exact integration-by-parts identity  $\langle \psi | R_{\text{sym}} | \psi \rangle = -2 + \frac{1}{2} |\psi(1)|^2 / \langle \psi | \psi \rangle$  (correct), measured  $|\psi(1)|^2 \sim 10^{-9}$  on its eigenstates, and concluded the boundary term vanishes, hence  $R_{\text{sym}} = -2I$  exactly — an “exact rigid shift”,  $\zeta$  the cleanest sector. But  $|\psi(1)|^2 \sim 10^{-9}$  is an *artifact of the Dirichlet construction*, not a property of the corpus APS eigenstates. **We build the corpus APS operator and measure the zeta-residual on it.** The APS radial eigenfunctions are  $R_{nl}(r) = r^{-1} J_{l+1}(kr)$  with  $k = 2n + l + 2$  (the APS quantisation), reproducing the integer ladder  $(2n + l + 2)^2$  and distinct from the Dirichlet squared-Bessel spectrum; they do *not* vanish at the wall,  $R_{nl}(1) = J_{l+1}(2n + l + 2) \neq 0$ . The exact identity is verified directly on these eigenstates (the numeric  $\langle R | r \partial_r | R \rangle / \langle R | R \rangle$  matches  $-2 + \frac{1}{2} R(1)^2 / \langle R | R \rangle$  to  $10^{-5}$ ). **The decisive measurement.** Under the corpus APS BC the boundary term is *large and state-dependent* (range  $0.018 \dots 2.238$ , mean  $0.79$ , not  $\sim 10^{-9}$ ), and  $\langle \psi_n | R_{\text{sym}} | \psi_n \rangle$  ranges from  $-1.98$  to  $+0.24$  across modes — emphatically not the rigid  $-2$ , and not even sign-definite as a shift (the  $l = 2, n = 0$  mode gives  $+0.24$ ). The state-dependence (std  $0.70$ ) is  $\sim 420\times$  A331's moat  $1.67 \times 10^{-3}$ . So A333's exact rigid shift is a **Dirichlet artifact**: the boundary term A333 measured as  $\sim 10^{-9}$  and killed is the *live* term under the corpus boundary condition. The decoupling  $\Phi = (\text{rigid} - 2\zeta) \oplus (\text{shape})$  *fails* (the zeta diagonal is mode-structured, cosine with the uniform vector  $0.864$ , not  $1.0000$ );  $\zeta$  is genuinely the soft, state-dependent sector — the original A332 obstruction, not cleaned. The consequence for (iii) is honest: A333's HIT does not transfer; free-weight injectivity rests on A331's in-box numerical scan with a *real* analytic gap at the zeta sector under the corpus operator, and the corpus tie  $\zeta = \alpha^{5/4}$  remains the clean closure for the corpus-posed family. Verifier `verify_P334.py`: 16 PASS / 0 FAIL. No published number changes.

# 1 The probe and the discipline

This is the corpus-faithfulness fix for the whole sub-result (iii) thread. The cross-thread inconsistency it resolves is real and was flagged on verification of A333: Heart 2 sub-result (i) settled the boundary condition on APS (A329, from P18 lines 393–396), while sub-result (iii) silently used Dirichlet. The binding solution-bar discipline governs: do not force closure, do not manufacture a fan-out, honesty over closure. Here that discipline is the substance — the honest discharge of the probe is to build the operator the corpus actually specifies and report what the zeta sector does on *it*, even though the answer refutes A333’s HIT. The brief-authoring discipline of re-grounding against the canonical source (here A329 and P18 lines 393–396, not the convenient interior mesh) is exactly what was missed across the (iii) thread and is corrected here.

The operator is P18’s bulk Dirac<sup>2</sup> radial form on the unit ball, the same one A329 fixed:  $-d_r^2 - (3/r)d_r + l(l+2)/r^2$  on the  $r^3$  measure, with regularity at  $r = 0$  and APS spectral projection at  $r = 1$  (P18 lines 393–396). Sub-result (iii) is a question about *this* operator; the Dirichlet model the (iii) thread used is a different operator with a different spectrum.

## 2 The corpus APS operator, built and validated

The radial equation  $[-d_r^2 - (3/r)d_r + l(l+2)/r^2]R = ER$  is solved by the regular root  $R(r) = r^{-1}J_{l+1}(\sqrt{E}r)$  (the singular root  $r^{-1}Y_{l+1}$  is not in  $L^2(r^3 dr)$ , A329 check 9). The quantisation is what distinguishes the two boundary conditions, and it is the whole point.

**Proposition 2.1** (The corpus APS spectrum and its eigenstates). *Under the APS spectral projection at  $r = 1$  the admissible radial wavenumbers are the integer ladder  $k = 2n + l + 2$  (boundary Dirac base  $l + 2$ , the integerised  $|D_{S^3}| = l + 3/2$ , plus radial overtone  $2n$  in steps of two; A329 check 12), so the spectrum is  $E_{nl} = (2n + l + 2)^2$ :  $l = 0$  gives 4, 16, 36, 64;  $l = 1$  gives 9, 25, 49, 81. This is exactly P18 line 171. The eigenfunctions  $R_{nl}(r) = r^{-1}J_{l+1}((2n + l + 2)r)$  do not vanish at the wall:  $R_{nl}(1) = J_{l+1}(2n + l + 2)$ , e.g. for  $l = 0$  the values are (0.577, −0.066, −0.277, 0.235) —  $O(10^{-2} \dots 10^{-1})$ , not the  $\sim 10^{-9}$  A333 measured on the Dirichlet model.*

The verifier reproduces this from scratch (checks 3–6): the integer ladder, the APS quantisation structure, the nonzero boundary values, and the finite positive  $L^2(r^3 dr)$  norms that make the residual a well-defined expectation. The Bessel implementation (numpy-only series, the same as A329) is cross-checked against known  $J_1$  zeros and against a finite-difference derivative (checks 1–2). The guardrail of the whole probe is that the APS spectrum *must* come out  $(2n + l + 2)^2$  and *must* differ from the Dirichlet squared-Bessel zeros; both are verified (checks 7–8). If the spectrum had come out as squared Bessel zeros we would be back on Dirichlet — the failure mode this probe exists to escape.

## 3 The exact identity, verified on the APS eigenstates

A333’s integration-by-parts decomposition is correct and is reproduced here. Under the measure  $w = r^3$ , for the first-order generator  $R = r\partial_r$ ,

$$\langle \psi | R_{\text{sym}} | \psi \rangle_w = -2 + \frac{1}{2} \frac{|\psi(1)|^2}{\langle \psi | \psi \rangle_w},$$

a state-independent bulk shift  $-2$  plus the APS boundary term. What A333 got wrong is not the identity but the value of  $\psi(1)$ . We verify the identity *directly* on the corpus APS eigenstates: the

numeric  $\langle R|r\partial_r|R\rangle_w/\langle R|R\rangle_w$ , computed from the eigenfunction and its analytic derivative with no  $-2$  assumed, equals  $-2 + \frac{1}{2}R(1)^2/\langle R|R\rangle_w$  to  $1.1 \times 10^{-5}$  across all modes (check 9), and the residual is mesh-converged (Richardson-stable to  $10^{-5}$  under refinement, check 10) — these are continuum values, not discretisation.

## 4 The decisive measurement: zeta is soft under APS

**Proposition 4.1** (The zeta-residual under the corpus boundary condition). *On the corpus APS eigenstates the boundary term  $\frac{1}{2}|\psi(1)|^2/\langle\psi|\psi\rangle$  is large and state-dependent: it ranges from 0.018 (the  $l=0, n=1$  mode,  $k=4$ ) to 2.238 (the  $l=1, n=0$  mode,  $k=3$ ), with mean 0.79 — not the  $\sim 10^{-9}$  A333 found on the Dirichlet model. Consequently  $\langle\psi_n|R_{\text{sym}}|\psi_n\rangle$  ranges from  $-1.98$  to  $+0.24$  across modes (for  $l=0$ :  $-0.69, -1.97, -1.32, -1.26$ ), emphatically not the rigid  $-2$ , and not even sign-definite as a shift (the  $l=2, n=0$  mode gives  $+0.24$ ). The state-dependence (standard deviation across modes) is 0.70, roughly  $420\times$  A331’s moat  $1.67 \times 10^{-3}$ , and the zeta diagonal is mode-structured (cosine with the uniform vector 0.864, not 1.0000).*

This is the measurement A333 never ran on the right operator (checks 11–15). It inverts A333’s finding term by term. A333: boundary term  $\sim 10^{-9}$  (empirically zero); APS: boundary term  $0.02 \dots 2.2$  (large, state-dependent). A333:  $R_{\text{sym}} = -2$  exactly for all  $n$  (rigid shift); APS:  $R_{\text{sym}} \in [-1.98, +0.24]$  across  $n$  (state-dependent, not even a fixed-sign shift). A333: residue  $\rightarrow 0$  in the continuum (discretisation); APS: residual is  $O(1)$ , mesh-converged, far above the moat. The boundary term A333 killed —  $\psi(1) = J_{l+1}(2n+l+2) \neq 0$  — is precisely the live term the brief hypothesised, and it is *not* refuted under the corpus boundary condition.

## 5 The verdict and what it does to (iii)

**Remark 5.1** (The Dirichlet rigid-shift picture does not transfer). *Under the corpus APS boundary condition the zeta sector is genuinely soft and state-dependent:  $\langle\psi_n|R_{\text{sym}}|\psi_n\rangle$  varies by  $O(1)$  across modes,  $\sim 420\times$  A331’s moat, and the rigid-offset decoupling  $\Phi = (\text{rigid} - 2\zeta) \oplus (\text{shape})$  fails. A333’s exact rigid shift holds only for the Dirichlet model. So A332’s lone non-sign-definite sector — the zeta obstruction — is real under the corpus operator, not cleaned; A333’s HIT does not transfer. Sub-result (iii) is less closed than A333 implied: free-weight injectivity rests on A331’s in-box numerical scan (itself a Dirichlet model, also requiring an APS redo) with a genuine analytic gap at the zeta sector under the corpus operator. The corpus tie  $\zeta = \alpha^{5/4}$  (P18 line 348) remains the clean closure for the corpus-posed family; the free-weight analytic global proof stays open, and is harder than A333 implied.*

## 6 Gate ledger

| Object                                             | Gate               | Finding                                                                                         |
|----------------------------------------------------|--------------------|-------------------------------------------------------------------------------------------------|
| Corpus APS operator built                          | DONE               | $(2n + l + 2)^2$ spectrum, $\psi(1) \neq 0$ , distinct from                                     |
| Exact IBP identity on APS states                   | DERIVED            | $\langle R r\partial_r R\rangle/\langle R R\rangle = -2 + \frac{1}{2}R(1)^2/\langle R R\rangle$ |
| APS boundary term                                  | LIVE               | large & state-dependent ( $0.02 \dots 2.2$ ), not $\sim 10$                                     |
| $\langle \psi_n   R_{\text{sym}}   \psi_n \rangle$ | STATE-DEP          | ranges $-1.98 \dots +0.24$ , not the rigid $-2$                                                 |
| A333 exact rigid shift                             | DIRICHLET ARTIFACT | does not transfer to the corpus operator                                                        |
| Zeta-residual vs A331 moat                         | ABOVE MOAT         | std $0.70 \approx 420\times$ moat; threatens injectivity                                        |
| (iii) free-weight injectivity                      | OPEN (corpus BC)   | real analytic gap at zeta under APS; tie clean                                                  |

## 7 Pre-registered fan-out

The probe decisively settles the orientation question: under the corpus BC the zeta sector is soft, so the (iii) injectivity results must be re-established on the APS operator, not transferred from the Dirichlet model. Two faithful fan-out probes follow; we do not manufacture beyond them.

1. **A335 — the bounded-box second-preimage scan on the APS operator.** Redo A331's free-weight scan (the  $\alpha$ - $\zeta$  ray with  $\gamma, \beta$  re-optimised, the moat, the zeta-valley floor) on the corpus APS operator built here, where  $\psi(1) \neq 0$  and the boundary residual is live. The question is whether the in-box global result (no second preimage, canonical the unique machine-zero) survives the boundary term, or whether the genuinely soft zeta sector opens a second preimage the Dirichlet model hid.
2. **A336 — the Hellmann–Feynman monotonicity on the APS operator.** Redo A332's structural-monotonicity argument on the APS operator. The three shape sectors ( $\alpha\rho$ ,  $\beta M$ ,  $\gamma \text{Re } \omega^k$ ) are multiplication/projection/character operators whose sign-definiteness should be BC-independent, but this must be re-verified on the APS eigenstates; and the zeta diagonal, now state-dependent rather than near-uniform, must be re-characterised to determine whether per-axis monotonicity in  $\zeta$  survives at all under the corpus BC or whether the soft sector breaks it outright.

If A335 finds no second preimage and A336 re-establishes the three shape sectors, (iii) stands as a corpus-faithful in-box numerical injectivity result with the zeta sector an honest soft direction (the tie the clean closure), genuinely parallel to where A331/A332 left the Dirichlet model — but now on the right operator. If A335 finds a second preimage, the free-weight version is non-injective under the corpus BC and the tie is load-bearing, not merely clean. We report whichever is true.

## 8 Conclusion

The probe builds the corpus APS radial operator the whole sub-result (iii) thread should have used —  $R_{nl}(r) = r^{-1}J_{l+1}((2n + l + 2)r)$ , spectrum  $(2n + l + 2)^2$  (P18 line 171), eigenstates that do not vanish at the wall — validates it against the integer ladder and against its distinctness from the Dirichlet squared-Bessel spectrum, and measures the zeta-sector residual on it. The exact integration-by-parts identity  $\langle \psi | R_{\text{sym}} | \psi \rangle = -2 + \frac{1}{2}|\psi(1)|^2/\langle \psi | \psi \rangle$  (A333's, correct) is verified directly on the APS eigenstates. Under the corpus boundary condition the boundary term is large and state-dependent ( $0.02 \dots 2.2$ , not  $\sim 10^{-9}$ ), and  $\langle \psi_n | R_{\text{sym}} | \psi_n \rangle$  ranges from  $-1.98$  to  $+0.24$  across modes — not the rigid  $-2$ , not sign-definite, with  $O(1)$  state-dependence some  $420\times$  A331's moat. So A333's exact rigid shift is a Dirichlet artifact that does not transfer: the boundary term A333

killed is the live term,  $\zeta$  is genuinely the soft, state-dependent sector under the corpus operator, and the rigid-offset decoupling fails. Sub-result (iii) is therefore less closed than A333 implied: its free-weight injectivity has a real analytic gap at the zeta sector under the corpus boundary condition, A331's in-box scan and A332's monotonicity must be redone on the APS operator (A335/A336 pre-registered), and the corpus tie  $\zeta = \alpha^{5/4}$  remains the clean closure for the corpus-posed family. The cross-thread inconsistency — Heart 2 (i) on APS (A329) versus (iii) on Dirichlet — is resolved in favour of APS: the corpus operator carries the APS BC, and (iii) must be argued on it. A294's re-typing of the whole P18-T2 stands; sub-results (i) (A328/A329) and (ii) (A310) are unchanged. No canon source files are edited; no published number changes; the  $\hat{O}$  assembly and its derived observables are untouched. Verifier `verify_P334.py` reaches 16 PASS / 0 FAIL and exits zero.