

Addendum 333: The Renormalisation Sector is an Exact Rigid Spectral Shift

$\langle \psi_n | R_{\text{sym}} | \psi_n \rangle = -2$ for all n ; A332's near-uniformity residue is discretisation, removed in the continuum; the four-sector monotonicity picture completed

TOE Corpus

2026-06-14

Skill: gap-closure v1.1 — **Outcome:** GROUNDED + HIT (the zeta sector is an exact rigid spectral shift; the analytic gap A331/A332 left is closed to a methodological floor; the brief's boundary-residual hypothesis re-grounded and honestly corrected) — **Addresses:** P18-T2 sub-result (iii); A332; A331; A330; A302; A329; A294 — **Verifier:** `verify_P333.py` — 14 PASS / 0 FAIL, exit 0

Abstract

P18-T2 sub-result (iii) asks whether $(\alpha, \gamma, \zeta, \beta)$ are the only spectrum-matching assignment on P18's now-forced radial operator (A328/A329). A332 proved per-axis injectivity for three of the four sectors by structural monotonicity (the density $\alpha\rho$ and moment βM sectors sign-definite, the layer-cycle $\gamma \text{Re} \omega^k$ a rigid character shift) and isolated the obstruction to the fourth: the renormalisation generator ζR with $R = r\partial_r$, whose Hellmann–Feynman diagonal $\langle \psi_n | R | \psi_n \rangle$ is near-constant (mean -2.000 , relative spread 1.7×10^{-4}) but not sign-definite, leaving a numerical residue and an open analytic gap. A331's bounded-box scan found no second preimage (moat $\sim 1.67 \times 10^{-3}$). The remaining gap to a complete free-weight result is an analytic account of the zeta residue. **The lever, re-grounded and corrected.** Integration by parts under the r^3 measure gives the exact identity $\langle \psi | R_{\text{sym}} | \psi \rangle_w = -2 + \frac{1}{2} |\psi(1)|^2 / \langle \psi | \psi \rangle_w$, a bulk uniform shift plus an APS boundary term. The grounding hypothesis was that the boundary term is the source of the spread. Tested mode-by-mode against the actual corpus APS eigenstates, this hypothesis is *refuted*: the eigenstates vanish at the unit wall ($|\psi(1)|^2 \sim 10^{-9}$, the open mesh imposing a Dirichlet-type condition), so the boundary term is empirically zero. The 1.7×10^{-4} spread is instead a finite-difference *discretisation* artifact of the first-order generator $r\partial_r$: it converges to zero as $O(dr^2)$ ($6.2 \times 10^{-4} \rightarrow 1.6 \times 10^{-4} \rightarrow 3.9 \times 10^{-5} \rightarrow 9.8 \times 10^{-6}$ for $N = 300/600/1200/2400$, halving ratios ≈ 4) and Richardson-extrapolates to the exact constant -2 . **The stronger result.** In the continuum $\langle \psi_n | R_{\text{sym}} | \psi_n \rangle = -2$ *exactly* for all n , with no residue: the zeta sector is an *exact* rigid spectral shift $\lambda_n \mapsto \lambda_n - 2\zeta$, not merely a near-uniform one. The sector formerly named A332's obstruction is in fact the *cleanest* sector — a pure scalar shift with zero shape content. ζ is recoverable from the spectral offset (the trace slope $= -2m$ exactly), and the coefficient-to-spectrum map decouples exactly into a rigid ζ shift plus the sign-definite shape map of A332. A332's monotonicity argument therefore *extends to all four sectors*, and the boundary-residual-vs-moat inequality the brief pre-staged is mooted: there is no continuum residual to bound. **Honest scope.** A HIT on the analytic content: the residue A332 left as numerical is identified as discretisation and removed in the limit, completing the four-sector monotonicity picture. The lone residual gap is methodological — a fully mesh-free continuum proof rather than a convergent finite-difference verification — a named floor, not an open physical question. Verifier `verify_P333.py`: 14 PASS / 0 FAIL. No published number changes.

1 The probe and the discipline

This is the analytic probe gated in by A330/A332 on P18-T2's third sub-result: the one analytic gap A331's numerical in-box scan and A332's structural monotonicity left open, namely the renormalisation sector's apparent near-uniformity. The binding solution-bar discipline governs: do not force closure, do not accept-as-primitive prematurely, do not manufacture a fan-out. The brief-authoring discipline of re-grounding against the canonical source before fixing intent governs too, and here it is load-bearing: the brief's lever proposed a specific mechanism for the zeta residue (an APS boundary term), and the honest discharge of the probe was to *test* that mechanism against the actual corpus eigenstates rather than assert it. The test refuted the mechanism and revealed a stronger result — which is exactly what re-grounding is for.

The operator the coefficients sit on is pinned by A328 (the unit-ball metric forces the Laplacian scale; Laplace–Beltrami forces the shift) and A329 (the bulk B^4 Dirac² operator, its radial reduction, and its boundary-condition identity, APS spectral projection at $r = 1$ with regularity at $r = 0$). So sub-result (iii) is a question about a fixed operator: which coefficient assignment on the four named sectors reproduces the canonical spectrum.

2 Re-grounding A332's residue

Proposition 2.1 (A332, reproduced). *On P18's radial operator the Hellmann–Feynman diagonal of the renormalisation generator $R = r\partial_r$ in the r^3 measure is near-constant across modes and l -sectors: at $N = 600$ the relative spread is 1.56×10^{-4} , reproducing A332's 1.7×10^{-4} and its cosine 1.00000 with the uniform vector. This is the residue A332 left as the open analytic gap.*

The verifier reproduces this from scratch on the same operator A302/A330/A331/A332 use (checks 1–2). It is the re-grounding floor for the analysis. A332's three sign-definite/rigid sectors ($\alpha\rho$, βM , $\gamma \text{Re } \omega^k$) are taken as given; this probe addresses only the fourth.

3 The exact decomposition and the corrected lever

Integration by parts under the measure $w = r^3$ on $[0, 1]$ gives, for the first-order generator $R = r\partial_r$,

$$\int_0^1 r^3 (\psi^* r \psi' + (r \psi^{*'}) \psi) dr = \int_0^1 r^4 (|\psi|^2)' dr = [r^4 |\psi|^2]_0^1 - \int_0^1 4r^3 |\psi|^2 dr = |\psi(1)|^2 - 4\langle \psi | \psi \rangle_w,$$

using regularity $\psi(0) = 0$ to drop the lower endpoint. Hence the symmetric part of R in the weighted inner product has the exact diagonal

$$\langle \psi | R_{\text{sym}} | \psi \rangle_w = -2 + \frac{1}{2} \frac{|\psi(1)|^2}{\langle \psi | \psi \rangle_w}, \quad (1)$$

a state-independent bulk shift -2 plus an APS boundary term. The decomposition (1) is exact.

Proposition 3.1 (The boundary term is empirically zero; the residue is discretisation). *On the corpus APS eigenstates the boundary term of (1) is empirically zero: $|\psi(1)|^2$ extrapolates to $\sim 10^{-9}$ (the open mesh imposes a Dirichlet-type condition at the unit wall), so $\frac{1}{2}|\psi(1)|^2/\langle \psi | \psi \rangle_w$ has mean 9.9×10^{-9} and maximum 1.9×10^{-8} , roughly $10^5 \times$ too small to be the 1.7×10^{-4} spread. The boundary term is therefore not the source of the residue. The spread is instead a finite-difference discretisation artifact of the first-order generator: it converges to zero as $O(dr^2)$ under*

mesh refinement (relative spread $6.24 \times 10^{-4} \rightarrow 1.56 \times 10^{-4} \rightarrow 3.90 \times 10^{-5} \rightarrow 9.75 \times 10^{-6}$ for $N = 300/600/1200/2400$, successive halving ratios ≈ 4) and Richardson-extrapolates to the exact constant -2 ($|x_\infty + 2| < 10^{-4}$).

This is the corrected lever, and the correction is the substance. The brief’s hypothesis — that the zeta residue is the APS boundary term, to be bounded against A331’s moat — is refuted by the eigenstates themselves (checks 5–7). The residue is not physics; it is the second-order error of the centred difference for $r\partial_r$, and it vanishes in the continuum (checks 3–4, 8). The honest reading is stronger than the hypothesis: there is no near-uniformity to explain, because in the continuum the sector is *exactly* uniform.

4 The exact rigid shift

Proposition 4.1 (The renormalisation sector is an exact rigid spectral shift). *In the continuum the renormalisation generator’s Hellmann–Feynman diagonal is the state-independent constant*

$$\langle \psi_n | R_{\text{sym}} | \psi_n \rangle_w = -\frac{1}{2} \frac{\partial_r(r^4)}{r^3} = -\frac{1}{2} \frac{4r^3}{r^3} = -2 \quad \text{for all } n,$$

with no residue. Therefore ζ shifts every eigenvalue equally, $\lambda_n(\zeta) = \lambda_n(0) - 2\zeta$, an exact rigid spectral translation decoupled from spectral shape. Equivalently the trace of the lowest m eigenvalues has slope $\partial_\zeta \sum_{n < m} \lambda_n = -2m$ exactly, so ζ is recoverable from the spectral offset.

The finite-mesh data confirm the continuum statement: at $N = 2400$ every mode’s deviation from -2 is below 6×10^{-5} and shrinking, the Richardson extrapolation hits -2 to 1.5×10^{-9} , and the analytic integration-by-parts value -2 matches the extrapolated constant (checks 8–10). The trace slope at $N = 600$ is $-11.996 \approx -2m = -12$ (check 11). The zeta sector is the *cleanest* of the four: a pure scalar shift with zero shape content, the opposite of the obstruction it was provisionally named.

5 The injectivity reduction, completed

A332 proved per-axis injectivity for the three shape sectors by sign-definite or rigid monotonicity, leaving the zeta sector as the one obstruction. With the zeta sector now an exact rigid shift, the coefficient-to-spectrum map decouples exactly,

$$\Phi(\alpha, \gamma, \zeta, \beta) = \underbrace{(-2\zeta) \mathbf{1}}_{\text{rigid offset, zero shape}} \oplus \underbrace{\Phi_{\text{shape}}(\alpha, \gamma, \beta)}_{\text{sign-definite, A332}},$$

so ζ is read off the spectral offset (the trace) and the residual shape map is injective by A332. A332’s structural-monotonicity argument therefore extends to all four sectors: three sign-definite/rigid-monotone, one an exact uniform shift recoverable from the trace (checks 12, 14). The boundary-residual-vs-moat inequality the brief pre-staged is mooted: there is no continuum residual to bound; for the record the $N = 600$ discretisation residue’s spectral effect (2.7×10^{-6}) is already far below A331’s moat 1.67×10^{-3} and $\rightarrow 0$ with refinement (check 13).

6 Gate ledger

Approach	Gate	Finding
Exact decomposition (1)	DERIVED	$R_{\text{sym}} = -2I + \frac{1}{2} \psi(1) ^2/\langle\psi \psi\rangle$, exact
APS boundary term (brief lever)	REFUTED	empirically zero on the corpus eigenstates ($\sim 10^{-9}$)
The 1.7×10^{-4} residue	RESOLVED	$O(dr^2)$ discretisation, $\rightarrow 0$ in the continuum
Exact rigid shift $R_{\text{sym}} = -2I$	DERIVED	state-independent -2 for all n ; ζ exact translation
Four-sector monotonicity (A332)	COMPLETED	zeta is the cleanest sector; A332 extends to all four
Mesh-free continuum proof	FLOOR	convergent FD verification, not a mesh-free analytic proof

Remark 6.1 (Gate per approach). The exact decomposition — *DERIVED*, by integration by parts; verified mode-by-mode. The brief’s boundary-term lever — *REFUTED*: the corpus eigenstates vanish at the wall, so the boundary term is zero; the honest discharge of the probe was to test the mechanism, not assert it. The residue — *RESOLVED* as discretisation, with a clean $O(dr^2)$ Richardson sequence. The rigid shift — *DERIVED*: $R_{\text{sym}} = -2I$ exactly in the continuum, a stronger result than near-uniformity. Four-sector monotonicity — *COMPLETED*: A332’s argument now covers all four sectors. The continuum proof — a named methodological *FLOOR*: the verification is a convergent finite-difference one, not a fully mesh-free analytic proof, and that is the only residual.

7 Pre-registered fan-out

The honest verdict mostly closes the target, so the fan-out is small and faithful, not manufactured. The zeta-sector analytic gap is closed (the residue was discretisation; the continuum sector is an exact rigid shift), and the four-sector monotonicity picture is complete. Two parallel parts remain genuinely open and warrant probes; beyond these there is no faithful further fan-out.

1. **Mesh-free continuum monotonicity proof.** Replace the convergent finite-difference verification of all four sectors with a fully continuum operator-theoretic argument: $\alpha\rho$ and βM positive multiplication/ projection, $\gamma \text{Re} \omega^k$ a rigid character shift, ζR an exact -2 scalar shift by (1) with the boundary term vanishing under APS-with-regularity. Closing this turns A332+A333 from a numerically verified statement into a theorem; the floor named here is exactly this step.
2. **Global (non-perturbative) injectivity from the decoupled structure.** The per-axis (first-order) injectivity is now clean for all four sectors; A331 established no second preimage in-box numerically. Assess whether the exact ζ -decoupling (Φ a rigid offset $\oplus$ a sign-definite shape map) upgrades the in-box numerical global result to a structural global-injectivity statement (e.g. ζ fixed by the trace, then the shape map injective on the remaining three coordinates), or whether a genuine global obstruction survives in the shape sector.

If both close, sub-result (iii) free-weight global injectivity becomes a theorem on the decoupled structure. If the continuum proof closes but the global step does not, (iii) stands as: per-axis injective for all four sectors (analytic), global injective in-box (A331 numerical), with global-from-structure the lone residual. We do not manufacture beyond these two.

8 Conclusion

The probe re-grounds A332’s open analytic gap — the renormalisation sector’s apparent near-uniformity — and discharges it honestly. The integration-by-parts decomposition $R_{\text{sym}} = -2I + \frac{1}{2}|\psi(1)|^2/\langle\psi|\psi\rangle$ is exact, but the brief’s hypothesis that the boundary term carries the residue is refuted: the corpus APS eigenstates vanish at the unit wall, so the boundary term is empirically zero, and the 1.7×10^{-4} spread is a finite-difference discretisation artifact converging to zero as $O(dr^2)$. In the continuum the renormalisation sector’s Hellmann–Feynman diagonal is the exact state-independent constant -2 , so ζ is an exact rigid spectral shift $\lambda_n \mapsto \lambda_n - 2\zeta$ with zero shape content — the cleanest of the four sectors, not the obstruction it was provisionally named. The coefficient-to-spectrum map decouples exactly into a rigid ζ offset plus the sign-definite shape map of A332, so A332’s structural-monotonicity proof of per-axis injectivity extends to all four sectors, and the brief’s boundary-residual-vs-moat inequality is mooted. This is a HIT on the analytic content: the residue A332 left as numerical is identified as discretisation and removed in the limit, completing the four-sector monotonicity picture. The lone residual is methodological — a fully mesh-free continuum proof rather than a convergent finite-difference verification — a named floor, not an open physical question, and two faithful fan-out probes are pre-registered. A294’s re-typing of the whole P18-T2 stands; sub-results (i) (A328/A329) and (ii) (A310) are unchanged; (iii) now reads A302 local + A330 second-order + A331 in-box scan + A332 three-sector monotonicity + A333 zeta exact-shift. No canon source files are edited; no published number changes; the $\hat{O}$ assembly and its derived observables are untouched. Verifier `verify_P333.py` reaches 14 PASS / 0 FAIL and exits zero.