

Addendum 332: Coefficient Injectivity by Structural Monotonicity

Three sectors are sign-definite-monotone; the obstruction to a clean global injectivity proof is isolated to the single renormalization direction

TOE Corpus

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Skill: gap-closure v1.1 — **Outcome:** PARTIAL (the second of A330’s two pre-registered fan-out parts; the structural-monotonicity complement to A331’s scan; injective in (α, γ, β) , the obstruction isolated to ζ) — **Addresses:** P18-T2 sub-result (iii); A330; A331; A302; A312 — **Verifier:** `verify_P332.py` — 16 PASS / 0 FAIL, exit 0

Abstract

P18-T2 sub-result (iii) asks whether the four coefficients $(\alpha, \gamma, \zeta, \beta)$ are the only spectrum-matching assignment on P18’s now-forced radial operator (A328/A329). A330 reproduced A302’s first-order local rank-4 result, upgraded it to a strict second-order local minimum, and located one genuine residual in the free-weight version: a near-flat ζ valley along the α - ζ near-degenerate ray, the soft direction. It pre-registered two parallel fan-out parts on the free-weight global injectivity question: a bounded-box second-preimage scan (A331, no second preimage in the box) and a structural monotonicity argument. This addendum is the second part. By Hellmann–Feynman, the per-coefficient spectral response $\partial\lambda_n/\partial c_i$ is the diagonal matrix element $\langle\psi_n|\text{Sector}_i|\psi_n\rangle$, and a sector gives a strictly monotone single-coefficient map (hence injective) iff that diagonal has a fixed sign. **Three of the four sectors qualify.** The density sector $\alpha\rho$ is multiplication by a nonnegative function ($\rho \geq 0$ on $[0, 1]$), so $\langle\psi|\rho|\psi\rangle > 0$ (116 to 181); the moment sector $\beta\mathcal{M}$ is a positive operator ($\mu_n \geq 0$, P_n projections), so $\langle\psi|\mathcal{M}|\psi\rangle \in [0, 1] \geq 0$; the layer-cycle sector $\gamma\text{Re}\omega^k$ contributes a rigid state-independent nonzero shift, monotone per $\mathbb{Z}_3$ character block. So the coefficient-to-spectrum map is injective in the (α, γ, β) directions. **The fourth is the obstruction.** The renormalization sector $\zeta\mathcal{R}$ with $\mathcal{R} = r\partial_r$ is a pure first-order generator (its operator diagonal is exactly zero, no multiplication part), and its Hellmann–Feynman diagonal $\langle\psi|\mathcal{R}|\psi\rangle$ is near-constant across all modes and l -sectors (mean -2.000 , relative spread 1.7×10^{-4} , cosine with the uniform vector 1.00000): ζ contributes a near-*uniform* spectral shift, the soft/degenerate direction. This single sector is the only obstruction to a clean global monotonicity proof, and it is exactly A330’s soft direction, A331’s ζ valley, and the freedom the corpus tie $\zeta = \alpha^{5/4}$ removes — one obstruction, three views. **Honest scope: PARTIAL.** Monotonicity proves injectivity for three sectors and isolates the obstruction to ζ ; it does not by itself give a full analytic global proof, because ζ breaks the fixed-sign multiplication structure the other three enjoy. The honest joint statement, combining with A331, is: injective in (α, γ, β) by structural monotonicity, no second preimage along the residual ζ direction in the admissible box, and the corpus tie as the clean closure for the corpus-posed family. Verifier `verify_P332.py`: 16 PASS / 0 FAIL. No published number changes.

1 The fan-out part and the discipline

This is the second of the two parallel fan-out parts A330 pre-registered on the free-weight global injectivity question. A331 ran the first, a bounded-box second-preimage scan, and found no second

preimage in the admissible box (the ζ valley floor strictly positive, the residual-minimizing α pinned at canonical, the moat bounded below by $\varepsilon > 10^{-4}$): a numerical, in-box result. This part runs the analytic complement. The two parts test the same residual — the free-weight ζ valley — from opposite sides: A331 by direct search, this part by asking whether the coefficient-to-spectrum map is *structurally* injective, a statement stronger than any scan where it holds.

The binding solution-bar discipline carried from Heart 1 governs: do not force closure, do not manufacture a faithful object, do not overclaim a structural isolation as a full analytic proof. The likely verdict is PARTIAL, and it is the honest one: monotonicity closes three of the four directions cleanly and isolates the entire obstruction to the one sector A330 and A331 both already named. That isolation, made structural rather than numerical, is the value of the probe.

2 The Hellmann–Feynman criterion

The four sectors enter P18’s radial operator additively with coefficients $(\alpha, \gamma, \zeta, \beta)$:

$$\hat{O}_{l,k} = -\frac{d^2}{dr^2} - \frac{3}{r} \frac{d}{dr} + \frac{l(l+2)}{r^2} + V_{\text{self}} + \alpha\rho + \gamma \operatorname{Re} \omega^k + \zeta r \partial_r + \beta \sum_n \frac{\mu_n}{\mu_0} P_n.$$

For a coefficient c_i multiplying a sector operator S_i , the Hellmann–Feynman theorem gives the first-order spectral response as the diagonal matrix element in the eigenbasis,

$$\frac{\partial \lambda_n}{\partial c_i} = \langle \psi_n | S_i | \psi_n \rangle,$$

computed in the corpus r^3 measure. The verifier confirms this identity against finite differences for α and ζ to 4×10^{-2} and 1×10^{-3} respectively (checks 1–2). A sector contributes *monotonically* to every eigenvalue if and only if its diagonal $\langle \psi_n | S_i | \psi_n \rangle$ has a fixed sign on the relevant subspace: then by Weyl monotonicity increasing c_i moves all eigenvalues one way, the single-coefficient map $c_i \mapsto \lambda(c_i)$ is strictly monotone, and a strictly monotone map is injective. We assess each sector against this criterion.

3 The three sign-definite sectors

Proposition 3.1 (Density: $\alpha\rho$ is sign-definite-monotone). *The density $\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x$ is nonnegative on $[0, 1]$ (minimum 0.011 on the interior mesh), so $\alpha\rho$ is multiplication by a nonnegative function and $\langle \psi | \rho | \psi \rangle > 0$ on every mode and l -sector (116 to 181). The single-coefficient map $\alpha \mapsto \lambda$ is strictly increasing, hence injective.*

Proposition 3.2 (Moment: $\beta\mathcal{M}$ is sign-definite-monotone). *The moment operator $\mathcal{M} = \sum_n (\mu_n/\mu_0) P_n$ has nonnegative coefficients ($\mu_n = \int_0^1 x^n \rho dx \geq 0$ because $\rho \geq 0$) and orthogonal projections P_n , so $\mathcal{M}$ is a positive operator and $\langle \psi | \mathcal{M} | \psi \rangle \in [0, 1] \geq 0$ on every mode. The single-coefficient map $\beta \mapsto \lambda$ is monotone, hence injective.*

Proposition 3.3 (Layer-cycle: $\gamma \operatorname{Re} \omega^k$ is rigid-monotone per character). *The layer cycle T_{cycle} has eigenvalues $1, \omega, \omega^2$; the real spectrum-matching reduction reads $\operatorname{Re} \omega^k = 1, -\frac{1}{2}, -\frac{1}{2}$. The response $\partial\lambda/\partial\gamma = \operatorname{Re} \omega^k$ is state-independent (a rigid additive shift) and nonzero in every block, so $\gamma \mapsto \lambda$ is strictly monotone — injective — within each character block. The sign splits by character (the trivial character raises, the two complex characters lower); it is not a single global sign, but it is a fixed nonzero shift per block, which is exactly what injectivity in γ requires once the character assignment is fixed.*

These are checks 3–8. Three of the four sectors give strictly monotone single-coefficient maps, so the coefficient-to-spectrum map is injective in the (α, γ, β) directions: no two distinct assignments differing only in α , only in γ , or only in β produce the same spectrum.

4 The renormalization sector is the obstruction

Proposition 4.1 (Renormalization: $\zeta\mathcal{R}$ is the soft, near-uniform sector). *The renormalization generator $\mathcal{R} = r\partial_r$ is a pure first-order operator: its matrix diagonal is exactly zero (no zeroth-order multiplication part), in contrast to ρ and $\mathcal{M}$, whose diagonals are nonnegative. So $\langle\psi|\mathcal{R}|\psi\rangle$ is not a fixed-sign multiplication readout. The decisive numerical fact is that $\langle\psi|\mathcal{R}|\psi\rangle$ is near-constant across all modes and l -sectors: mean -2.000 , relative spread 1.7×10^{-4} , and cosine with the uniform vector 1.00000 . Thus ζ contributes a near-uniform spectral shift. By contrast the density response is mode-structured (relative spread 0.087 , cosine with the uniform vector 0.996). The near-uniform value is not a numerical accident: the symmetric part of $r\partial_r$ in the r^3 measure is the state-independent constant*

$$\frac{1}{2}\langle\psi|r\partial_r + (r\partial_r)^\dagger|\psi\rangle = -\frac{1}{2}\frac{1}{r^3}\frac{d}{dr}(r^3 \cdot r) = -\frac{1}{2}\frac{4r^3}{r^3} = -2,$$

so to leading order $\langle\psi|\mathcal{R}|\psi\rangle$ is the exact constant -2 regardless of the eigenstate — a structural, not just numerical, reason the ζ shift is uniform (check 15).

A near-uniform shift is precisely the degenerate direction. A rigid shift of all eigenvalues by a common amount is what the other additive sectors can nearly absorb: it is collinear, to high precision, with the γ block shifts and with the flat part of the mode-structured α and β responses. This is the structural origin of the soft direction. The monotonicity argument that cleanly closes α , β , and γ does not give a fixed-sign multiplication readout for ζ that separates it from the rest; the near-uniformity is exactly the failure mode (checks 9–11).

5 The unification

The three views of the same obstruction now coincide:

- A330 found, in normalized coordinates, an α – ζ soft plane and, in raw coordinates, that the soft direction is almost exactly “move ζ alone” with ζ spectrally cheap.
- A331 found, by direct scan, a near-flat ζ valley whose floor stays strictly positive, with the residual-minimizing α pinned at canonical — no second preimage in the box.
- This addendum finds, by Hellmann–Feynman, that ζ ’s response is a near-uniform spectral shift: the analytic reason ζ is cheap, the valley is flat, and the soft plane exists.

A near-uniform Hellmann–Feynman diagonal is spectrally cheap (small structured response, hence A330’s $71\times$ magnitude gap), it makes the residual flat along ζ (hence A331’s shallow valley), and it is exactly the freedom the corpus tie $\zeta = \alpha^{5/4}$ removes by slaving ζ to the stiff, mode-structured α (A330’s $820\times$ tied response). One obstruction, three views (check 14).

6 Gate ledger

Sector	Hellmann–Feynman diagonal	Monotonicity verdict
$\alpha\rho$ (density)	$\langle\rho\rangle \in [116, 181] > 0$	MONOTONE (sign-definite)
$\beta\mathcal{M}$ (moment)	$\langle\mathcal{M}\rangle \in [0, 1] \geq 0$	MONOTONE (sign-definite)
$\gamma\text{Re}\omega^k$ (cycle)	rigid shift $1, -\frac{1}{2}, -\frac{1}{2}$	MONOTONE per character block
$\zeta r\partial_r$ (renorm)	near-uniform (-2.000 , spread 10^{-4})	NOT clean: the obstruction

Remark 6.1 (The assembled claim and its honest scope). *Three of the four sectors give strictly monotone single-coefficient maps, so the coefficient-to-spectrum map is injective in the (α, γ, β) directions, and the entire non-injectivity risk is confined to the single ζ direction. This is a structural result, stronger where it holds than any scan. But it is PARTIAL: it does not by itself give a full analytic global proof of the free-weight version, because ζ breaks the fixed-sign multiplication structure the other three enjoy — the near-uniformity is shallow, not a clean sign, and a near-uniform direction is exactly where a far-away ζ with re-optimized companions can nearly reproduce the spectrum. The ζ gap is closed only numerically and in-box by A331 (no second preimage), and analytically by the corpus tie (which removes the ζ freedom). The honest joint statement: injective in (α, γ, β) by structural monotonicity, plus no second preimage along the residual ζ direction in the admissible box (A331), equals free-weight global injectivity established to the honesty floor; a full analytic proof of the free-weight version stays open at the ζ sector, and the corpus tie $\zeta = \alpha^{5/4}$ is the clean closure for the corpus-posed family (checks 12–13, 16).*

7 Conclusion

The structural monotonicity argument closes three of the four coefficient directions cleanly. By Hellmann–Feynman the per-coefficient spectral response is the sector’s diagonal matrix element; the density $\alpha\rho$ and moment $\beta\mathcal{M}$ sectors are sign-definite (nonnegative diagonals), and the layer-cycle $\gamma\text{Re}\omega^k$ sector is a rigid nonzero shift monotone per $\mathbb{Z}_3$ character block, so the coefficient-to-spectrum map is injective in the (α, γ, β) directions. The renormalization sector $\zeta r\partial_r$ is a pure first-order generator whose Hellmann–Feynman diagonal is near-uniform across modes, a near-rigid spectral shift; it is the one sector that breaks the fixed-sign multiplication structure, and it is exactly A330’s soft direction, A331’s ζ valley, and the freedom the corpus tie removes. The entire non-injectivity risk is therefore isolated to the single ζ direction, structurally rather than numerically. We do not claim a full analytic global proof: the verdict is PARTIAL, and the ζ gap is closed in the admissible box by A331’s scan and analytically by the corpus tie. With A330 (local and second-order rigidity, the free-vs-tied resolution), A331 (no second preimage in the box), and this addendum (three sectors structurally injective, the obstruction isolated), free-weight global injectivity is established to the honesty floor and its residual is named precisely. A294’s re-typing of the whole P18-T2 stands; sub-results (i) (A328/A329) and (ii) (A310) are unchanged. No canon source files are edited; this is a standalone addendum. No published number changes; the $\hat{O}$ assembly and its derived observables are untouched. Verifier `verify_P332.py` reaches 16 PASS / 0 FAIL and exits zero.