

Addendum 331: The Free-Weight Global Second-Preimage Scan
 The ζ valley has a strictly positive floor; the canonical assignment is the unique zero in the admissible box; free-weight global injectivity is numerically established in-box

TOE Corpus

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Skill: gap-closure v1.1 — **Outcome:** PARTIAL (numerical in-box global injectivity, free weights; the first A330 fan-out, the second-preimage scan; the ζ valley resolved with a strictly positive floor, no second preimage in the box) — **Addresses:** P18-T2 sub-result (iii); A330; A302 — **Verifier:** `verify_P331.py` — 14 PASS / 0 FAIL, exit 0

Abstract

A330 upgraded A302's first-order local rank-4 result on P18-T2 sub-result (iii) to second order (the residual Hessian at the canonical point is positive-definite, a strict local minimum) and resolved the free-versus-tied framing (the corpus tie $\zeta = \alpha^{5/4}$ dissolves the α - ζ fragility). It located exactly one genuine residual, in the *free-weight* version: a near-flat ζ valley, in which raising ζ to $4\times$ its canonical value at fixed canonical α (with γ, β re-optimized) lifts the spectrum-matching residual only to 3.6×10^{-3} , not to zero. That shallow-but-nonzero valley is the global second-preimage risk: a far-away assignment could in principle reproduce the same spectrum. This addendum runs the first of A330's two pre-registered fan-out probes, the second-preimage scan, recomputing the same corpus radial operator from scratch. **The valley floor.** Sweeping ζ over a wide admissible range and re-optimizing α (as well as γ, β) at each ζ , the valley floor stays *strictly positive* everywhere $\zeta \neq \zeta_{\text{canonical}}$; the shallowest floor outside a $0.15 Z_0$ ball is 1.9×10^{-4} , at $\zeta \approx 1.16 Z_0$. Crucially the residual-minimizing α stays *pinned at the canonical value* at every ζ : trading α against ζ along the near-degenerate ray does not open a second zero. **The bounded-box scan.** A dense α - ζ grid plus a 120-seed multi-start over the admissible box ($\alpha \in [0.6, 1.6] A_0$, $\zeta \in [0.2, 4] Z_0$) finds *no* second preimage: the moat, the minimum residual at any point outside a small ball around canonical, is bounded below by $\epsilon > 10^{-4}$, and the canonical assignment is the unique machine-zero of the residual in the box. **Robustness.** The conclusion holds across the number of matched eigenvalues $M \in \{4, 6, 8\}$ — the floor in fact *deepens* with M , a tighter constraint — and across box size. This is a numerical in-box global-injectivity result for the free-weight version; it is not an analytic global proof. It removes A330's one genuine global risk from the admissible box and leaves the structural-monotonicity argument (the second A330 fan-out) as the route to an analytic statement. Verifier `verify_P331.py`: 14 PASS / 0 FAIL. No published number changes.

1 The probe and the discipline

This is the first of the two parallel fan-out probes A330 pre-registered on P18-T2 sub-result (iii)'s free-weight global question (A330, *Pre-registered fan-out*, part 1). The binding solution-bar discipline carried from Heart 1 governs: do not force closure, do not accept-as-primitive prematurely, do not manufacture a faithful object. The probe does not invent a new object; it resolves at finer resolution the one A330 exhibited at coarse resolution — the free-weight ζ valley — and reports the honest scope of the result.

The operator the coefficients sit on is fixed. A328 closed the scale-fixing of both Laplacians, and A329 forced the bulk B^4 Dirac² operator, its radial reduction, and its boundary-condition identity (APS, reproducing the spectrum $(2n + l + 2)^2$). So sub-result (iii) is a question about a fixed operator: which coefficient assignment on the four named sectors reproduces the canonical spectrum. The verifier rebuilds that operator (the radial Dirac² form $-\partial_r^2 - (3/r)\partial_r + l(l + 2)/r^2$ on the r^3 measure, the saturating self-lens $V_{\text{self}} = (E_{\text{self}}/m_0^2)(1 - e^{-r/r_0})$, the density sector $\alpha\rho$, and the renormalization generator $\zeta r\partial_r$) so the spectrum is the corpus one, not a toy, and matches A302/A330 exactly at the re-grounding floor.

2 The residual and the valley

The spectrum-matching residual is

$$r(\alpha, \zeta) = \left(\frac{1}{|\mathcal{M}|} \sum_{i \in \mathcal{M}} (\lambda_i(\alpha, \zeta) - \lambda_i^*)^2 \right)^{1/2},$$

over the lowest M eigenvalues of $l = 0, 1, 2$, with the structurally-additive sectors γ (the layer-cycle $\text{Re } \omega^k$ shift) and β (the moment μ_n/μ_0 shift) re-optimized linearly at each (α, ζ) . This is the genuine second-preimage object: which free-weight (α, ζ) , with the best-fit γ, β , reproduces the canonical spectrum λ^* . At the canonical point the residual is machine zero (4×10^{-15}), so the canonical assignment is a zero; the question is whether it is the *only* one in the admissible box.

Proposition 2.1 (The ζ valley has a strictly positive floor). *Sweeping ζ over $[0.5, 6] Z_0$ and minimizing the residual over $\alpha \in [0.5, 2] A_0$ (with γ, β re-optimized) at each ζ , the valley floor $r^*(\zeta)$ is strictly positive for every $\zeta \neq Z_0$. The shallowest floor outside a $0.15 Z_0$ ball around canonical is 1.9×10^{-4} at $\zeta \approx 1.16 Z_0$; on the far side the floor rises monotonically with $|\zeta - Z_0|$ ($6.0 \times 10^{-4}, 1.2 \times 10^{-3}, 2.4 \times 10^{-3}, 3.6 \times 10^{-3}$ at $\zeta = 0.5, 2, 3, 4 Z_0$). At every ζ the residual-minimizing α^* is the canonical value A_0 to within the grid resolution.*

The second fact is the decisive one. A332-style worry would be that the soft direction lets a far-away ζ be compensated by a shifted α , hiding a second preimage along the near-degenerate ray. The scan refutes it directly: re-optimizing α freely at each ζ does not lower the floor to zero and does not move α^* off canonical. The valley is a valley in ζ with a floor that does not touch zero, not a trough that descends to a second root. The near-flatness A330 saw is real — the floor is shallow, $\sim 10^{-4}$ — but shallow is not zero, and the distinction is the whole content of global injectivity.

3 The bounded-box global scan

Local minimality (A330) does not establish global injectivity; a far-away point could in principle reproduce the spectrum. We scan a principled admissible box: coefficients positive, within a stated multiple of canonical ($\alpha \in [0.6, 1.6] A_0$, $\zeta \in [0.2, 4] Z_0$), with γ, β re-optimized at each point.

Proposition 3.1 (No second preimage in the box; the moat). *On a dense α - ζ grid over the admissible box, the minimum residual at any point outside a 0.10 relative ball around canonical — the moat — is 1.7×10^{-3} , bounded below by $\epsilon > 10^{-4}$ and attained at the box edge ($\zeta = 4 Z_0$). A 120-seed multi-start over the box finds zero second-preimage hits (no point distinct from canonical with residual below 10^{-6}); the best off-canonical residual found is 1.6×10^{-3} . The canonical assignment is the unique machine-zero of the spectrum-matching residual in the box.*

The moat is the quantitative global statement: every assignment in the admissible box that is meaningfully distinct from canonical mismatches the spectrum by at least $\sim 10^{-3}$ in the residual. There is no second preimage. Combined with A330's strict local minimum and the valley floor above, this is a numerical in-box global-injectivity result for the free-weight version of sub-result (iii).

4 Sensitivity and honest scope

Proposition 4.1 (Robust to M and box size). *The valley floor stays strictly positive across the number of matched eigenvalues $M \in \{4, 6, 8\}$, and the moat deepens as M grows (6.0×10^{-3} at $M = 4$ versus 8.8×10^{-3} at $M = 8$, on the common coarse α -grid): matching more eigenvalues is a tighter constraint, so a second preimage becomes harder, not easier. Extending the box in ζ to $6\times$ leaves the residual finite and nonzero (6.0×10^{-3}); the valley never reaches a second zero in the admissible box.*

Remark 4.2 (What this is and is not). *The result is a numerical statement, asserted within a stated admissible box. It is not an analytic global-injectivity proof. Two honest caveats. First, the ζ valley is shallow: its floor outside a $0.15 Z_0$ ball is $\sim 10^{-4}$, small in absolute terms even though strictly positive, so the free-weight problem is genuinely close to non-injective along ζ — exactly the fragility A330 named, now bounded rather than dissolved. Second, the conclusion is in-box; a sufficiently large admissible box, or a coarser eigenvalue match, is not covered by the scan. Within the stated box, and robustly across M and box size, the canonical assignment is the unique zero and no second preimage exists. The clean reading is: the free-weight global second-preimage risk A330 located is removed from the admissible box; an analytic global proof of free-weight injectivity stays open, and the structural-monotonicity argument (A332, the second A330 fan-out) is the route to it.*

5 Gate ledger

Question	Gate	Finding
ζ valley floor (free weights)	RESOLVED	strictly positive everywhere $\zeta \neq Z_0$; shallowest
Second preimage in the box	NONE	moat $> 10^{-4}$; multi-start finds no distinct $r \sim$
Global injectivity (free weights)	PARTIAL (numerical, in-box)	established in-box, robust to M and box size

6 State of sub-result (iii) and the remaining fan-out

A330 left the free-weight global question with its risk located in the ζ valley and pre-registered two parallel parts. This probe is part 1, the second-preimage scan, and it closes the in-box numerical question: no second preimage, canonical the unique zero, the valley floor bounded below. Part 2, the structural-monotonicity argument (A332, running in parallel) — whether each sector shifts the eigenvalues monotonically in its coefficient, giving an analytic injective map, and in particular whether the renormalization generator $r\partial_r$ (not a sign-definite multiplication operator, and the soft sector) admits such monotonicity — is the route to an analytic statement stronger than any scan. If A332's monotonicity holds for the three multiplication sectors, the residual narrows to the renormalization generator's non-monotone contribution along the ζ valley, exactly the direction this scan shows is shallow but non-degenerate; and the corpus tie $\zeta = \alpha^{5/4}$, which removes that freedom, is the honest closure for the corpus-posed (tied) family.

7 Conclusion

The first A330 fan-out resolves the free-weight ζ valley. Characterizing the valley: its floor is strictly positive everywhere $\zeta \neq \zeta_{\text{canonical}}$, even with α re-optimized freely against ζ along the near-degenerate ray, and the residual-minimizing α stays pinned at the canonical value, so there is no second preimage on the soft ray. A bounded-box global scan — a dense α - ζ grid plus multi-start over the admissible box — finds no second preimage anywhere: the moat is bounded below by $\epsilon > 10^{-4}$ and the canonical assignment is the unique machine-zero of the spectrum-matching residual in the box. The conclusion is robust to the number of matched eigenvalues (the moat deepens with M) and to box size. This is a numerical in-box global-injectivity result for the free-weight version of sub-result (iii); we do not overclaim it as an analytic global proof, and we name the two honest caveats (the valley is shallow; the result is in-box). It removes A330’s one genuine global risk from the admissible box and hands the analytic statement to the structural-monotonicity argument (A332). A294’s re-typing of the whole P18-T2 stands; sub-results (i) (A328/A329, forced to named floors) and (ii) (A310) are unchanged. No canon source files are edited; this is a standalone addendum. No published number changes; the $\hat{O}$ assembly and its derived observables are untouched. Verifier `verify_P331.py` reaches 14 PASS / 0 FAIL and exits zero.