

Addendum 330: Coefficient Uniqueness, Global and Higher Order

The corpus tie $\zeta = \alpha^{5/4}$ dissolves the soft direction; the residual is a strict local minimum; free-weight global injectivity stays open

TOE Corpus

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Skill: gap-closure v1.1 — **Outcome:** GROUNDED + STEP (Heart 2 grounding probe for sub-result (iii); A302 reproduced and upgraded to second order; the free-vs-tied framing resolved, dissolving the α - ζ fragility; one fan-out pre-registered) — **Addresses:** P18-T2 sub-result (iii); A302; A328; A329; A294 — **Verifier:** `verify_P330.py` — 13 PASS / 0 FAIL, exit 0

Abstract

P18-T2 sub-result (iii) asks whether the four coefficients $(\alpha, \gamma, \zeta, \beta)$ are the only spectrum-matching assignment on P18's now-forced radial operator (A328/A329). A302 established the first-order *local* form: treating the four as free assembly weights, the spectrum-response Jacobian at the canonical point has full rank 4, so the coefficients are locally the unique spectrum-matching assignment; but the density coupling α and the renormalization generator ζ are $\sim 99.86\%$ spectrally collinear, the *soft direction* where uniqueness is fragile. A302 was explicitly silent on global injectivity and on second order. This addendum reproduces A302 from scratch and takes the first concrete steps on the global frontier, and the central step resolves a framing question that by itself changes the fragility reading. **Free versus tied.** A302 treated all four coefficients as free weights, the hardest version. P18 also fixes $\zeta = \alpha^{5/4}$ (lines 348, 358, 579, 608), a functional tie of ζ to α . We show that the soft direction, in raw coordinates, is almost exactly “move ζ alone” (the raw near-null direction of the α - ζ block is $(0.014, 0.99990)$), and ζ is spectrally cheap ($\|\partial\lambda/\partial\alpha\|$ is $71\times \|\partial\lambda/\partial\zeta\|$). The corpus tie forbids moving ζ alone: along $\zeta = \alpha^{5/4}$, $d\zeta/d\alpha = \frac{5}{4}\alpha^{1/4}$, so any motion in ζ drags α , which carries the large response. The tie direction is transverse to the flat direction ($|\text{overlap}| = 0.356$), and the tied single-direction spectral response is $\sim 820\times$ the free flat-direction response. The α - ζ soft *plane* that carries A302's fragility exists only in the free-weight version; under the corpus tie it collapses to a stiff 1-D curve. **Second order.** The residual Hessian at the canonical point is positive-definite (eigenvalues $3.20, 3.62, 54, 2.2 \times 10^6$), so the canonical point is a strict local minimum of the spectrum-matching residual, not merely a non-degenerate critical point; the softest curvature strengthens with sector coverage ($0.053 \rightarrow 0.845 \rightarrow 3.20$), mirroring A302's first order. **Global.** A coarse second-preimage scan exhibits the free-weight fragility directly: a near-flat ζ valley (raising ζ to $4\times$ lifts the residual only to 3.6×10^{-3}), while the tied curve is a sharp isolated well (α to $1.5\times/2\times$ lifts it to $0.14/0.28$). So sub-result (iii) for the corpus-tied family is far less fragile than A302's free-weight bound suggests; the free-weight global injectivity is the genuinely hard residual and the warranted fan-out. Global uniqueness stays open. Verifier `verify_P330.py`: 13 PASS / 0 FAIL. No published number changes.

1 The grounding probe and the discipline

This is the Heart 2 grounding/orientation probe (A322 pattern) for P18-T2's third sub-result, the global and higher-order coefficient-uniqueness question. The binding solution-bar discipline

carried from Heart 1 governs: do not force closure, do not accept-as-primitive prematurely, do not manufacture a faithful object to justify a fan-out. The probe establishes the framing and gates a fan-out; it need not close anything. The brief-authoring discipline of re-grounding against the canonical source before fixing intent governs too, and it earns its keep: the question of whether sub-result (iii) means “free weights” (A302’s version) or “the corpus-tied family” is exactly the kind of framing question that, read against P18 rather than the obvious default, dissolves a fragility that looked irreducible.

The operator the coefficients sit on is itself now pinned. A328 closed the scale-fixing of both Laplacians (the unit-ball metric forces the unit radius, Laplace–Beltrami forces the zero shift), and A329 forced the bulk B^4 Dirac² operator, its radial reduction, and its boundary-condition identity (APS, reproducing the spectrum $(2n + l + 2)^2$). So sub-result (iii) is a question about a fixed operator: which coefficient assignment on the four named sectors reproduces the canonical spectrum.

2 Re-grounding A302

Proposition 2.1 (A302, reproduced). *With the four named sectors carrying free weights on P18’s radial operator (density $\alpha\rho$, layer-cycle $\gamma\text{Re}\omega^k$, renormalization $\zeta r\partial_r$, moment $\beta\mu_n/\mu_0$), the column-normalized spectrum-response Jacobian at the canonical point $(\alpha, \gamma, \zeta, \beta) = (1/137.036, 3/4, \alpha^{5/4}, 3\pi/20)$ has four nonzero singular values with $\sigma_{\min}/\sigma_{\max} = 9.3 \times 10^{-3}$ on $l = 0$, rank exactly 4; the ratio strengthens to $2.3 \times 10^{-2} \rightarrow 3.5 \times 10^{-2}$ as $l \leq 1, 2$ are added; and the density and renormalization response columns are collinear to 0.9986, the soft direction.*

The verifier reproduces every number of A302 from scratch (checks 1–3), using A302’s exact operator construction so the spectrum is the corpus one. This is the re-grounding floor for everything that follows.

3 The free-versus-tied resolution

The decisive question the probe must answer is what sub-result (iii) means. P18 line 376 states that “the four coefficients $(\alpha, \gamma, \zeta, \beta)$ are pinned by the references cited in the Coefficient Determination proposition”, each independently: α from $\int \rho = \alpha^{-1}$ (P03), $\gamma = 3/4$ from five-fold convergence (P10), $\zeta = \alpha^{5/4}$ from dimensional analysis (P16), $\beta = 3\pi/20$ from gravity (P17). The strongest result is therefore global uniqueness of the *free-weight* version, which is A302’s framing and the hardest one. But P18 also fixes $\zeta = \alpha^{5/4}$ as a *functional* tie of ζ to α (lines 348, 358, 579, 608), and whether that tie is part of the admissible assembly changes the soft direction.

Proposition 3.1 (The corpus tie removes the soft plane). *The soft direction of A302, in raw (α, ζ) coordinates, is almost exactly “move ζ alone”: the raw near-null right-singular vector of the α – ζ response block is $(0.014, 0.99990)$. The renormalization sector is spectrally cheap, $\|\partial\lambda/\partial\alpha\|/\|\partial\lambda/\partial\zeta\| = 71.3$, so moving ζ alone barely perturbs the spectrum, which is the fragility. The corpus tie $\zeta = \alpha^{5/4}$ forbids this motion: along the tie $d\zeta/d\alpha = \frac{5}{4}\alpha^{1/4}$, so the admissible direction is $(1, \frac{5}{4}\alpha^{1/4})$, which is transverse to the soft direction ($|\text{overlap}| = 0.356$), and the spectral response along the tied curve is $\sim 820\times$ the response along the free flat direction. The α – ζ soft plane collapses to a stiff 1-D curve.*

This is the central finding, and it is a clean resolution rather than a manufactured one. A302’s 99.86% collinearity is a statement about the *shape* of the two normalized response columns; in raw magnitude the two sectors are wildly different, α stiff and ζ soft. The fragility lives entirely in the

freedom to vary ζ independently of α . The corpus tie removes exactly that freedom. So the honest reading of A302’s soft direction is: it is the worst case of the *free-weight* problem, and the corpus does not pose the free-weight problem for (α, ζ) — it ties them. For the corpus-tied family the α – ζ fragility is dissolved (checks 4–7).

4 Second order: a strict local minimum

A302 was first order, silent on second-order coincidences. We extend to the Hessian. The spectrum-matching residual is $R(c) = \sum_i (\lambda_i(c) - \lambda_i(c^*))^2$; at the canonical point c^* the residual and its gradient vanish, and the Hessian is exactly $2J^\top J$ (Gauss–Newton, exact at a zero-residual root). Positive-definiteness upgrades A302’s non-degenerate critical point to a strict local minimum.

Proposition 4.1 (Second-order rigidity). *The free-weight residual Hessian $2J^\top J$ at the canonical point is positive-definite, with eigenvalues 3.20, 3.62, 54, 2.19×10^6 ; the canonical assignment is a strict local minimum of the spectrum-matching residual. The softest curvature strengthens with sector coverage, $0.053 \rightarrow 0.845 \rightarrow 3.20$ for $l \leq 0, 1, 2$, mirroring A302’s first-order strengthening. The tied residual Hessian (coordinates α, γ, β , with ζ slaved) is also positive-definite, and the α – ζ soft plane is absent because ζ is no longer an independent coordinate.*

So sub-result (iii) is upgraded from first-order local to second-order local: the canonical point is a genuine strict minimum, and the strengthening-with-coverage structure A302 found at first order persists at second (checks 8–10).

5 Global: the free-weight valley and the tied well

Local and second-order rigidity do not establish global injectivity: a far-away point could in principle reproduce the same spectrum. The risk is concentrated along the soft ζ direction. A coarse second-preimage scan over a bounded admissible box, with γ, β re-optimized at each (α, ζ) , exhibits the free-weight fragility directly and the tie’s cure.

Proposition 5.1 (Free-weight valley, tied well). *At the canonical point the residual is machine zero (8×10^{-16}), so the canonical assignment is the global minimum. In the free-weight problem the ζ direction is a near-flat valley: raising ζ to $4\times$ its canonical value (at fixed canonical α, γ, β re-optimized) lifts the residual only to 3.6×10^{-3} . Along the tied curve $\zeta = \alpha^{5/4}$ the same problem is a sharp isolated well: raising α to $1.5 \times / 2\times$ lifts the residual to 0.14/0.28, more than $10\times$ the free ζ valley. The global second-preimage risk lives in the free-weight ζ valley, and the corpus tie removes it.*

This is the global statement matching the local one. The free-weight problem has a shallow valley along ζ where a far-away ζ with re-optimized γ, β nearly reproduces the spectrum — the genuine global fragility, and the reason a clean global-injectivity proof of the *free-weight* version is non-trivial. The corpus-tied problem replaces the valley with a sharp well (checks 11–13). We do not claim global injectivity is closed; we exhibit exactly where the free-weight risk lives and that the tie removes it.

6 Gate ledger

Approach	Gate	Finding
Free-vs-tied (the soft direction)	RESOLVED	tie $\zeta = \alpha^{5/4}$ removes the α - ζ soft plane; fragility dissolves
Second-order (Hessian) rigidity	DERIVED	residual Hessian positive-definite (strict local min); strengthened
Global injectivity (free weights)	PARTIAL / fan-out	free-weight ζ valley is the genuine global risk; second-order

Remark 6.1 (Gate per approach). Free-vs-tied — *RESOLVED*, and machine-derivable: the corpus tie is a P18 identity, and imposing it dissolves the α - ζ fragility (the soft plane becomes a stiff curve, tied response $\sim 820\times$ the free flat response). This is the high-value finding of the probe. Second order — *DERIVED*: the residual Hessian is positive-definite, a strict local minimum, strengthening with coverage; A302 is upgraded from first to second order, no fan-out needed. Global injectivity — *PARTIAL*: local and second-order rigidity plus the dissolved tie make the global question far less fragile than A302’s free-weight bound, but global injectivity of the free-weight version is not established; the free-weight ζ valley is the residual to control, and it warrants a fan-out.

7 Pre-registered fan-out

One fan-out is warranted, on the free-weight global question, with two parallel parts that test the same residual from opposite sides:

1. **Global second-preimage scan, free weights.** A finer bounded-box scan focused on the α - ζ near-degenerate ray with γ, β re-optimized, matching the lowest M eigenvalues, testing for a second $(\alpha, \gamma, \zeta, \beta)$ that reproduces the target spectrum. A clean “no second preimage in the admissible box” is a real (numerical) global result; the ζ valley exhibited here at coarse resolution is the object to resolve.
2. **Structural monotonicity argument.** Each sector enters the operator as a positive multiplication or first-order generator; if each shifts the eigenvalues monotonically in its coefficient, the coefficient-to-spectrum map is monotone hence injective, a structural global result stronger than any scan. Assess whether the ζ sector (the renormalization generator $r\partial_r$, which is not a sign-definite multiplication operator) admits such a monotonicity, since that is precisely the soft sector.

If the scan finds no second preimage and the monotonicity holds for the three multiplication sectors, the residual narrows to the renormalization generator’s non-monotone contribution along the ζ valley — and the corpus tie, which removes that freedom, is the honest closure for the corpus-posed (tied) family. If neither closes the free-weight version, sub-result (iii) for free weights stays open with its risk precisely located, which is itself a faithful result.

8 Conclusion

The grounding probe reproduces A302’s first-order local rank-4 result from scratch and advances the third sub-result on three fronts. The free-versus-tied framing is resolved: the α - ζ soft plane that carries A302’s fragility exists only in the free-weight version, and the corpus tie $\zeta = \alpha^{5/4}$ (P18 line 348) collapses it to a stiff curve, dissolving the fragility for the corpus-posed family. The second-order question is answered: the residual Hessian is positive-definite, so the canonical point is a strict local minimum, and the rigidity strengthens with sector coverage, upgrading A302 from first to second order. The global question is framed and its risk located: the free-weight problem has

a near-flat ζ valley, the genuine global fragility, while the tied curve is a sharp well. We pre-register one fan-out on the free-weight global injectivity (a finer second-preimage scan and a structural monotonicity argument), and we do not manufacture more. Global uniqueness of the free-weight version stays open with its risk precisely named; for the corpus-tied family the fragility is dissolved. A294's re-typing of the whole P18-T2 stands; sub-results (i) (A328/A329, forced to named floors) and (ii) (A310) are unchanged. No canon source files are edited; this is a standalone addendum. No published number changes; the $\hat{O}$ assembly and its derived observables are untouched. Verifier `verify_P330.py` reaches 13 PASS / 0 FAIL and exits zero.