

Addendum 328: The Sub-result (i) Frontier

Scale-fixing closes the Laplacians; the bulk Dirac² is partial; the multiplication-sector formulas are a definitional floor

TOE Corpus

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Skill: gap-closure v1.1 — **Outcome:** GROUNDED + STEP (ITEM 1 closed; ITEM 2 partial; ITEM 3 floor; fan-out: one probe) — **Addresses:** P18-T2 sub-result (i); A314; A312; A304; A327 — **Verifier:** `verify_P328.py` — all PASS

Abstract

P18-T2 sub-result (i) asks whether each sector operator of $\hat{O}$ is the only operator in its category. Five sectors are machine-verified forced: the layer cycle T_{cycle} (A304), the renormalization generator $R = r d/dr + \Delta_\psi$ including the additive shift (A312), the fiber Laplacian Δ_{S^1} (A312), the boundary Laplacian Δ_{S^3} up to scale (A314), and the bulk Dirac² Lichnerowicz constant $R/4 = 3/2$ on the round S^3 (A314). Three items remain open. This addendum, the Heart 2 grounding probe, re-grounds each against its canonical verifier and gates it, then takes the first concrete step on the crispest. **Item 1**, the scale-fixing of the Laplacians, closes: the unit ball $|x| \leq 1$ (P18) fixes the unit-radius round metric, whose Laplace–Beltrami operator has the spectrum $-L(L+2)$ with multiplier exactly one, so $a = 1$ is a corpus metric fact, not a convention; and the corpus condition that the boundary operator is the pure Laplace–Beltrami operator of the round metric, equivalently that it annihilates constants, fixes $b = 0$. So Δ_{S^3} and, by the identical argument, Δ_{S^1} go from unique up to $a \text{ Lap} + b I$ to fully forced. **Item 2**, the full bulk B^4 Dirac² operator, is partial: the Dirac operator is the unique first-order operator with principal symbol Clifford multiplication, machine-verified through the dimension-four Clifford relation, so D^2 is forced once D is; but the full B^4 boundary-value-problem spectrum is not reduced to a single spectral identity the way the round S^3 slice was. **Item 3**, the multiplication-sector formula-uniqueness of V_{self} and $\mathcal{M}$, is a definitional floor: re-grounded against P18, V_{self} is the saturating exponential $(E_{\text{self}}/m_0^2)(1 - e^{-r/r_0})$, not the quartic the probe brief carried (that is the OI-287-1 numerical production operator); given the formulas both are forced as multiplication and projection operators, but the formulas themselves are P18 posits, parallel to ρ and $G1$ (A327). The pre-registered fan-out is a single probe on Item 2; Item 1 closes here, Item 3 is a floor. Verifier `verify_P328.py`: 14 PASS / 0 FAIL.

1 The frontier and the discipline

A294 re-typed P18-T2 into three sub-results: (i) each sector operator is the only operator in its category; (ii) additive composition is the only admissible composition; (iii) the coefficients are the only spectrum-matching assignment. Sub-result (i) has advanced through A304, A312, and A314 to five machine-verified forced sectors. Three items remain, and the binding solution-bar discipline carried over from Heart 1 governs this probe: do not force closure, do not accept-as-primitive prematurely, and do not manufacture a faithful object to justify a fan-out. This addendum establishes the framing and gates the three items; it is not required to close anything, and it reports each gate honestly.

The geometry is P18's unit ball: $M = B^4 = \{x \in \mathbb{R}^4 : |x| \leq 1\}$ with boundary $\partial M = S^3$, the metric $ds^2 = dr^2 + r^2 d\Omega_3^2$ where $d\Omega_3^2$ is the round metric on S^3 (P18 §2.1). The bulk operator is $D_{B^4}^2 = -\Delta_{B^4} + R/4$ (P18 §3.1, the Lichnerowicz form).

2 Item 1: scale-fixing of the Laplacians (CLOSED)

2.1 Re-grounding

A314 forced Δ_{S^3} only up to the affine family $a \text{Lap} + bI$: the $\mathfrak{so}(4)$ -invariant second-order operator space is two-dimensional, spanned by $\{I, C_2\}$ with C_2 the $\mathfrak{so}(4)$ Casimir, and A314 identified C_2 with $-\Delta_{S^3}$ through $4j(j+1) = L(L+2)$ on the symmetric (j, j) irreps. A312 forced Δ_{S^1} only up to scale by the same isometry-group argument. The open question is whether a and b are fixed by a corpus normalization or remain a convention the corpus sets by fiat. This is the framing-audit question for the item: is the normalization a genuine corpus metric fact or a convenient fiat?

2.2 Gate: machine-derivable. The scale is a corpus metric fact

The answer is that the normalization is a metric fact, and the item closes. The Laplace–Beltrami operator of a fixed Riemannian metric is unique; there is no free multiplier once the metric is fixed. The metric is fixed by the unit ball: $|x| \leq 1$ forces the boundary to be the unit-radius round S^3 , whose Laplace–Beltrami spectrum is exactly $-L(L+2)$, with multiplier one. We verify (checks 1–4) that the $\mathfrak{so}(4)$ Casimir reproduces $L(L+2)$ for $L = 0, \dots, 4$ as $0, 3, 8, 15, 24$ with multiplier exactly one, that the metric-rescaling law $g \mapsto c^2 g$ sends the spectrum to $L(L+2)/c^2$ so that the unit ball $c = 1$ fixes $a = 1$, that a free scale is observable (a value $a = 2$ gives a spectrum $2L(L+2)$ distinguishable from $L(L+2)$ for every $L \geq 1$, so the scale is not a hidden convention), and that the identical argument fixes $a = 1$ for Δ_{S^1} on the unit Hopf fiber.

The shift b is fixed by the corpus condition that the boundary operator is the pure Laplace–Beltrami operator of the round metric (P18 lines 113–114), which is equivalent to the condition that it annihilates constants. We verify (checks 5–7) that the Laplacian annihilates the constant $L = 0$ mode exactly, that an affine $a \text{Lap} + bI$ with $b \neq 0$ leaves b on the constant mode and so fails the annihilation condition, forcing $b = 0$, and that $b = 0$ is equivalent to the operator having no zeroth-order piece, i.e. to its being the pure Laplace–Beltrami operator.

Proposition 2.1 (Δ_{S^3} and Δ_{S^1} fully forced). *On the unit ball geometry of P18, the boundary Laplacian Δ_{S^3} and the fiber Laplacian Δ_{S^1} are fully forced: $a = 1$ by the unit-radius round metric, $b = 0$ by the pure-Laplace–Beltrami / annihilates-constants condition. The affine freedom A314 and A312 left open is removed. This is a derivation, not a fiat: the metric is corpus-fixed, and the Laplace–Beltrami operator of a fixed metric is unique.*

This advances sub-result (i) from five sectors forced (two Laplacians only up to scale) to seven sector facts forced, with the two Laplacian scales now closed.

3 Item 2: full bulk B^4 Dirac² operator (PARTIAL)

3.1 Re-grounding

A314 checked only the round- S^3 spectral identity, the Lichnerowicz constant $R/4 = 3/2$. The full claim is that $D_{B^4}^2 = -\Delta_{B^4} + R/4$ (P18 line 151, the Weitzenböck/Lichnerowicz formula) is the unique second-order operator in the bulk Dirac category on the four-ball with its spin structure.

3.2 Gate: partial. Clifford-uniqueness is machine-checkable; the full BVP spectrum is not

The Dirac operator is the unique first-order operator with principal symbol equal to Clifford multiplication; this is what forces D , and D^2 is forced once D is. We verify (checks 8–9) the dimension-four Euclidean Clifford relation $\{\gamma_i, \gamma_j\} = 2\delta_{ij}I$ on an explicit chiral-basis spin representation, and that each γ_i is Hermitian and squares to the identity, confirming D is the genuine spin-Dirac operator of the metric and spin connection rather than an ad hoc first-order operator. The Weitzenböck identity $D^2 = \nabla^*\nabla + R/4$ then follows structurally.

We further verify (check 10) that the four-ball adds a genuinely new datum over the round- S^3 slice A314 already did: the flat interior has $R = 0$ so the curvature term $R/4$ vanishes in the bulk, while the round S^3 boundary has $R = 6$ so $R/4 = 3/2$ there. The bulk curvature term is carried entirely by the boundary, consistent with the radial form (P18 line 154) having no constant interior potential beyond $R/4$.

Remark 3.1 (Gate: PARTIAL). *The Clifford-uniqueness of D , hence of D^2 , is machine-verified representation theory, and the Weitzenböck form is a structural identity. What is not reduced to a single spectral identity, the way the round S^3 was, is the full B^4 boundary-value-problem spectrum. The structural content is forced; the full BVP spectrum is the residual (check 11).*

4 Item 3: multiplication-sector formula-uniqueness (DEFINITIONAL FLOOR)

4.1 Re-grounding, and a correction

A312 noted that V_{self} , ρ , and $\mathcal{M}$ are forced as multiplication or projection operators given their explicit formulas, but that the formula-uniqueness is open. ρ 's formula-uniqueness is OI-287-1 (Heart 1), which A327 showed bottoms out in a flagged posit. We must re-ground V_{self} and $\mathcal{M}$ against P18 directly, and here the canonical source corrects a memo claim: P18 line 216 gives

$$V_{\text{self}}^{\text{can}}(r) = \frac{E_{\text{self}}}{m_0^2}(1 - e^{-r/r_0}), \quad E_{\text{self}} = 13.177,$$

a saturating exponential with $V_{\text{self}}(0) = 0$ (regularity at the origin, P18 line 228), *not* the quartic $E_{\text{self}} x^2(1 - x)^2$ that the probe brief carried. That quartic is the numerical production operator of the OI-287-1 thread, a different object; the two forms differ (check 12). The moment operator is $\mathcal{M} = \sum_{n \geq 0} (\mu_n/\mu_0) P_n$ (P18 line 315), a spectral projector sum weighted by the corpus moments.

4.2 Gate: definitional floor

Given the formulas, both sectors are forced as multiplication and projection operators: V_{self} is diagonal in the position basis, hence commutes with ρ (check 13), and $\mathcal{M}$ is the projector sum $\sum_n w_n P_n$ in the eigenbasis (check 14). That much A312 already had. The formulas themselves, however, are P18 posits: V_{self} 's constants E_{self} , r_0 , m_0 are cited values, and $\mathcal{M}$'s weights μ_n/μ_0 are corpus moments (P02, P17). Neither is produced by a uniqueness argument within P18.

Remark 4.1 (Gate: DEFINITIONAL FLOOR). *V_{self} and $\mathcal{M}$ bottom out in “P18 states the formula”, exactly as ρ does (OI-287-1; A327 showed its magnitude is an irreducible flagged posit). This is a definitional floor, not a derivation, and naming it honestly is the result. We do not force these to look closable; they are floors, parallel to ρ and G1.*

5 The gate ledger and the pre-registered fan-out

Item	Gate	First step / status
1. Laplacian scale/shift	MACHINE-DERIVABLE	CLOSED here ($a = 1$, $b = 0$)
2. Full bulk B^4 Dirac ²	PARTIAL	Clifford-uniqueness verified; BVP residual
3. V_{self} / $\mathcal{M}$ formulas	DEFINITIONAL FLOOR	P18 posits, parallel to $\rho/G1$

Item 1 closes in this probe: the metric fixes the scale and the pure-Laplace–Beltrami condition fixes the shift, a derivation not a fiat. Item 3 is a definitional floor; there is no derivation to fan out to, only a posit to name, which we have done. Item 2 carries genuine machine-derivable content beyond what was checked: the Clifford-uniqueness of D is verified here, but the full B^4 boundary-value-problem spectrum, with the spin structure and the boundary condition at S^3 , is not reduced to a single spectral identity the way the round S^3 slice was.

Proposition 5.1 (Pre-registered fan-out: one probe). *The fan-out is a single probe on Item 2: construct the radial B^4 Dirac² boundary-value problem (P18 line 154, $-\partial_r^2 - (3/r)\partial_r + L_{S^3}^2/r^2 + R/4$) with the spin structure and a self-adjoint boundary condition at $r = 1$, and test machine-side whether its spectrum is forced by the Weitzenböck decomposition together with the boundary Laplacian (now fully forced by Item 1). If the BVP spectrum reduces to the bulk $\nabla^*\nabla$ spectrum plus the (vanishing interior) curvature term, the operator is forced and Item 2 closes; if a boundary-condition freedom survives, that freedom is the residual to gate. Item 1 needs no probe (closed here); Item 3 needs no probe (a floor).*

This is a deliberately small fan-out, consistent with the no-manufacture discipline: only Item 2 has machine-derivable content distinct from what is already established, so only Item 2 is fanned out.

6 Conclusion

Sub-result (i) of P18-T2 advances. Item 1, the scale-fixing of the Laplacians, closes: the unit ball fixes the metric scale $a = 1$ as a corpus fact and the pure-Laplace–Beltrami condition fixes the shift $b = 0$, so Δ_{S^3} and Δ_{S^1} are fully forced. Item 2, the full bulk B^4 Dirac², is partial: Clifford-uniqueness of D (hence D^2) is machine-verified, the full B^4 BVP spectrum is the residual, and a single probe is pre-registered. Item 3, the multiplication-sector formulas V_{self} and $\mathcal{M}$, is a definitional floor: P18 posits the formulas, parallel to ρ and $G1$, and the honest result is to name the floor. A294’s re-typing of the whole P18-T2 stands; sub-result (ii) (the A310 chain) and sub-result (iii) (A302) are unchanged. No canon source files are edited; this is a standalone addendum. Verifier `verify_P328.py` reaches 14 PASS / 0 FAIL and exits zero.