

Addendum 323: The $\mathbb{Z}_3$ Selection Rule

Whether the layer-cycle character grades the density's degrees, and what the forced pairing does and does not reach

TOE Corpus

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Skill: gap-closure v1.1 — **Outcome:** PARTIAL (a structural selection rule; the forced pairing is a cyclic shift of the assignment) — **Addresses:** OI-287-1 (corrected); A322 — **Verifier:** `verify_P323.py` — all PASS

Abstract

A322 gated one corpus-faithful object for the corrected OI-287-1: the three charged-lepton families as the ω^k eigenspaces of the layer-cycle operator T_{cycle} , a representation-theoretic grading rather than an expectation value. This probe tests, at that grading level and forming no overlap, the candidate reason for the family-to-coefficient assignment: whether the layer-cycle $\mathbb{Z}_3$ grades the degree- k Taylor term of the density ρ with character $k \bmod 3$, so that Schur selection forces each family to couple to a single degree. The corpus action is the forward fiber rotation $\varphi \mapsto \varphi + 2\pi/3$ of P18 §3.5, which by the Hopf relation sends the complex base coordinate $z \mapsto \omega z$ and so the degree- k monomial $x^k \mapsto \omega^k x^k$; the degree- k term of ρ therefore carries character $k \bmod 3$. The Schur sum $\frac{1}{3} \sum_j \omega^{(a-b)j} = \delta_{a \equiv b}$ then forces each family to exactly one degree. The selection rule is structurally exact. The forced pairing, however, is a cyclic shift of the mass-law assignment: the degrees present $\{1, 2, 3\}$ carry characters $\{1, 2, 0\}$, so degree three lands on character zero, the trivial family, not on level three, and with the family level $n = k + 1$ (the family characters of P06 and P20 are 0, 1, 2 while the mass-law levels are 1, 2, 3) the character pairing sends $(e, \mu, \tau) \mapsto (16, 2, 3)$, the cyclic shift of the corpus $(2, 3, 16)$. The verdict is PARTIAL: the index matching is a $\mathbb{Z}_3$ selection rule at the character level, the genuine mechanism the corrected problem sought, but its agreement with the mass law's $n \leftrightarrow n$ labelling depends on the level-character offset $n = k + 1$, which the corpus does not yet fix. The overlap-diagonal stays withdrawn; the residual passes to P3.

1 The Character Grading and the Selection Rule

Proposition 1.1 (The layer-cycle grades the density's degrees, and Schur selection forces one degree per family). *Let $\omega = e^{2\pi i/3}$ and let T_{cycle} be the layer-cycle generator, with $T_{\text{cycle}}^3 = I$ and eigenvalues the cube roots of unity, the three families being its ω^k eigenspaces, $k = 0, 1, 2$ (P18 §3.5, P20 Family Origin; the generator is forced by the $L(3, 1)$ orientation, A304). The corpus action of T_{cycle} is the forward fiber rotation $\varphi \mapsto \varphi + 2\pi/3$ on the Hopf fiber coordinate (P18 §3.5, proposition on the canonical selection of T_{cycle}). Under the Hopf fibration $S^3 \subset \mathbb{C}^2$ the fiber phase rotates the complex base coordinate, $z \mapsto e^{i\varphi} z$, so the forward generator sends $z \mapsto \omega z$ and the degree- k monomial transforms as $x^k \mapsto \omega^k x^k$. The degree- k Taylor term of $\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x$ therefore carries $\mathbb{Z}_3$ character $k \bmod 3$: the degrees present, 1, 2, 3, carry characters 1, 2, 0.*

For a family of character a and a density term of character b , the $\mathbb{Z}_3$ -invariant pairing is

$$P(a, b) = \frac{1}{3} \sum_{j=0}^2 \omega^{(a-b)j} = \delta_{a \equiv b \pmod{3}},$$

by Schur orthogonality. A coupling between a family and a density term is therefore nonzero only when their characters agree, and each family is forced to couple to exactly one of the three degrees, the one whose character matches. The index matching is a $\mathbb{Z}_3$ selection rule, not a coincidence: family of character k couples only to the degree of character k .

2 The Forced Pairing and the Offset It Leaves

Remark 2.1 (A cyclic shift, the level-character offset, and where the residual goes). *The selection rule fixes which degree each family couples to by character, but the resulting pairing is a cyclic shift of the assignment the mass law uses. The family characters are $e \mapsto 0$, $\mu \mapsto 1$, $\tau \mapsto 2$ (P06 §3, P20 Family Origin), while the mass-law levels are $e \mapsto 1$, $\mu \mapsto 2$, $\tau \mapsto 3$, so the level and the character of a family differ by one, $n = k + 1$. Matching by character then sends the character-zero family to the degree-three term, since degree three carries character zero, and gives $(e, \mu, \tau) \mapsto$ coefficients $(16, 2, 3)$. The corpus mass law assigns $(e, \mu, \tau) \mapsto (2, 3, 16)$. The two are related by a cyclic shift, not the identity: the selection rule is exact at the character level, but it does not reproduce the literal $n \leftrightarrow n$ pairing, because degree three lands on character zero rather than on level three.*

This is the honest content of the PARTIAL verdict. The candidate reason is real: the family-to-coefficient assignment is a $\mathbb{Z}_3$ selection rule, the degree- k term of ρ carrying character $k \bmod 3$ and Schur orthogonality forcing one degree per family. What it does not yet settle is the labelling, the offset $n = k + 1$ between the family level the mass law indexes and the character the grading assigns, so that the trivial family pairs with the cubic term rather than with level three. The corpus states the family characters and states the mass-law levels but does not fix their alignment, which is what would turn the cyclic shift into the identity. The overlap-diagonal of A299 through A320 remains withdrawn and is not revived here; no overlap or expectation value is formed, only the character of each degree and the Schur selection sum. The residual passes to P3: whether the identity $c_n = \rho^{(n)}(0)/(\pi^n n!)$ is itself a reading of the $\mathbb{Z}_3$ -graded structure of ρ on the lens space $L(3, 1)$, which would fix the level-character alignment from the geometry rather than leaving it to be assumed. P4 stays conditional on a faithful pairing with the labelling settled.

Status. OI-287-1 remains open in its corrected A287 and A296 form. The probe establishes a $\mathbb{Z}_3$ selection rule as the mechanism of the family-to-coefficient assignment at the character level, a derivation step on the gated object, and identifies the standing residual as the level-character offset $n = k + 1$ that the corpus does not yet fix. No published number changes; the A287 lepton law and its 10^{-4} accuracy are untouched.

Verifier. `verify_P323.py`: recomputes the $\mathbb{Z}_3$ structure ($T_{\text{cycle}}^3 = I$, eigenvalues $\{1, \omega, \omega^2\}$, cite A304); the Schur identity $\frac{1}{3} \sum_j \omega^{(a-b)j} = \delta_{a \equiv b}$ and that it forces one degree per family; the character of each density degree, $1, 2, 3 \mapsto 1, 2, 0$, under the corpus action $x^k \mapsto \omega^k x^k$; the forced pairing $(e, \mu, \tau) \mapsto (16, 2, 3)$ against the corpus $(2, 3, 16)$ and that it is a cyclic shift, not the identity; and the PARTIAL verdict's defining condition, that selection is structurally exact while the forced pairing is the cyclic reindex of the assignment with $n = k + 1$ and degree three on character zero. Nine checks, all PASS.