

Addendum 320: The Coupled Bulk-and-Boundary B^4 Problem

The arena that keeps both the radial bound state and the boundary harmonic overshoots the magnitude and does not reach the gap either reduction missed

TOE Corpus

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Skill: gap-closure v1.1 — **Outcome:** Case C (null; the coupled bulk-and-boundary reduction keeps both sectors, breaks the family degeneracy, but overshoots the magnitude past the gap the single reductions also missed, and does not improve on the standing fit) — **Addresses:** OI-287-1 — **Verifier:** `verify_P320.py` — all PASS

Abstract

Across seven arenas the magnitude entry of the charged-lepton selection rule splits in two: the channels that do not distinguish the three families undershoot, the pure-radial $\mu = 1.222$ and the r^3 measure $\mu = 1.337$, and the channels that give three distinct families overshoot, the fiber 2.3, the Dirac 3.14, the boundary-lens 3.86, and the boundary-harmonic 6.28. The target $\mu = 1.5$ sits in a gap, $[1.34, 2.0]$, reached by neither the radial reduction of Addenda 300–318 nor the pure-boundary reduction of Addendum 319. This addendum builds the one arena not yet tested, the coupled bulk-and-boundary B^4 problem. Paper 18 §4.3 gives the explicit radial form of the master operator, $\hat{O}_{l,k} = -d^2/dr^2 - (3/r)d/dr + l(l+2)/r^2 + V_{\text{eff}}(r) + \omega^k\gamma$, which carries both the boundary-harmonic centrifugal term $l(l+2)/r^2$ and the layer-cycle family term $\omega^k\gamma$ together. A state is the radial bound state of the $l = L$ sector times the boundary harmonic of degree L ; the coupled eigenvalue is the radial eigenvalue plus the boundary Casimir $L(L+2)$ plus the family phase $\text{Re}(\omega^k)\gamma$. The hypothesis was interpolation: the undershooting radial sector and the overshooting boundary sector together might land in the gap. The finding is null. The canonical coupled map, family n to boundary degree $L = n - 1$ coupled to its radial ground state, returns $\mu = 7.77$ with $\tau = 152$, far above the gap and above every prior overshoot; the linked map, family n the $(n - 1)$ -th radial mode of the $l = n - 1$ sector, returns $\mu = 3.90$. Both overshoot past 2.0, the canonical one hardest of all, because the boundary multiplicity factor $(L + 1)^2 = 1, 4, 9$ compounds the overshoot rather than tempering it. The two controls bracket the gap, pure radial at $\mu = 1.222$ and pure boundary at $\mu = 3.38$, yet no coupled map lands inside it. The magnitude is in none of the radial, the boundary, or their coupling.

1 The Probe

Proposition 1.1 (The coupled reduction overshoots the magnitude and does not reach the gap). *On the 4-ball $B^4 = (\text{radial } r \in [0, 1]) \times (\text{boundary } S^3)$ the master operator in the sector with fixed boundary angular momentum l and family index k is, by Paper 18 §4.3,*

$$\hat{O}_{l,k} = -\frac{d^2}{dr^2} - \frac{3}{r}\frac{d}{dr} + \frac{l(l+2)}{r^2} + V_{\text{eff}}(r) + \omega^k\gamma, \quad \gamma = \frac{3}{4}, \quad \omega = e^{2\pi i/3},$$

with V_{eff} the canonical effective potential held verbatim from the radial thread. This is the coupled object: the same radial operator the radial reduction used, now carrying both the boundary-harmonic

centrifugal term $l(l+2)/r^2$, the S^3 sector entering the radial equation, and the layer-cycle family term $\omega^k\gamma$, the $\mathbb{Z}_3$ sector. A state is $\psi_{n,L}(r)Y_L(S^3)$, the radial bound state of the $l = L$ sector times the boundary harmonic of degree L ; the radial reduction set $l = 0$ and dropped $\omega^k\gamma$, and the pure-boundary reduction kept $L(L+2)$ and dropped the radial bound state. The coupled eigenvalue is $\lambda_{n,L} = \varepsilon_{n,L}^{\text{rad}} + L(L+2) + \text{Re}(\omega^k)\gamma$, the radial eigenvalue plus the boundary Casimir plus the family phase, and the full B^4 overlap is the bulk density moment in the r^3 radial measure times the boundary harmonic multiplicity $(L+1)^2$, divided by $\lambda_{n,L}$. Two coupled maps are declared in advance. In the canonical map, family n is the boundary degree $L_n = n - 1$ coupled to the radial ground state of the $l = L_n$ sector with family index $k = n - 1$, the corpus $\hat{O}_{l,k}$ structure; the normalized diagonal is $(1, 7.77, 152.3)$ with $\mu = 7.77$ and $\tau = 152.3$, at distance 144.5 to the target $(1, 1.5, 8)$. In the linked map, the radial node index and the boundary degree are linked so that family n is the $(n - 1)$ -th radial mode of the $l = n - 1$ sector; the normalized diagonal is $(1, 3.90, 50.5)$ with $\mu = 3.90$, at distance 42.6. Each map produces three distinct entries, since the coupled sectors are genuinely distinct, but each overshoots, both μ values lying above the target and above the gap $[1.34, 2.0]$, and neither restores a valid column-standardized peak against the box control. The two controls reproduce the two limits the coupling was to interpolate between: the pure-radial control, $L = 0$ for all families, returns $\mu = 1.222$, recovering the radial undershoot, and the pure-boundary control, the radial ground fixed with the boundary Casimir and multiplicity carrying the distinction, returns $\mu = 3.38$, recovering the boundary overshoot. The controls bracket the gap, yet no coupled map lands inside it; neither improves on the standing-best distance 0.399 at $\mu = 1.222$.

Remark 1.2 (The verdict is null; the coupling does not interpolate). *The construction is the corpus coupled object, Paper 18 §4.3's own $\hat{O}_{l,k}$, built without fitting: the boundary Casimir $L(L+2) = 0, 3, 8$ and multiplicity $(L+1)^2 = 1, 4, 9$ are the genuine $\mathfrak{so}(4)$ data of Addendum 314, the radial bound states are the same ones the radial thread used, and the family phase $\omega^k\gamma$ carries $\gamma = 3/4$ from Paper 18 §4.2. The canonical map is declared up front as the corpus $\hat{O}_{l,k}$ pairing, family n to boundary degree $L = n - 1$ coupled to its radial ground; a tuned pairing would be a fit, and none is invoked. The hypothesis was interpolation, that the undershooting radial sector and the overshooting boundary sector together would land in the gap the single reductions missed. The coupling does the opposite. The canonical map overshoots to $\mu = 7.77$, hardest of any arena tested, because the boundary multiplicity factor $(L+1)^2$ that distinguishes the families is itself the quantity that drives the overshoot: it enters the overlap multiplicatively as 1, 4, 9 across the three families, and the radial coupling does not damp it. The linked map overshoots more gently, to $\mu = 3.90$, but still past 2.0. The two controls confirm the limits, pure radial at 1.222 below the gap and pure boundary at 3.38 above it, and confirm that the coupled problem inherits the boundary overshoot rather than tempering it with the radial undershoot. The seven-arena pattern holds one level further out: the structure that distinguishes the families overshoots the magnitude, now in the arena built precisely to interpolate. The reading is that the magnitude is in none of the radial reduction, the pure-boundary reduction, or their coupling. What this leaves is the framing. Three reductions and their union all place the magnitude outside reach, with the channels that distinguish the families overshooting and the channels that do not undershooting, and nothing landing between. That is the signature of a quantity computed as the wrong functional rather than in the wrong arena. Whether the target $(2, 3, 16)$ is an overlap-diagonal quantity at all is the question the companion probe takes up. The magnitude piece of OI-287-1 does not close here.*

Status. OI-287-1 remains open. A320 builds the coupled bulk-and-boundary B^4 problem, the arena that keeps both the radial bound state and the boundary harmonic, in the corpus form $\hat{O}_{l,k}$ of Paper 18 §4.3, with two coupled maps declared in advance: the canonical map, family n to

boundary degree $L = n - 1$ coupled to its radial ground, and the linked map, family n the $(n - 1)$ -th radial mode of the $l = n - 1$ sector. Both break the family degeneracy and give three distinct entries, but both overshoot the magnitude: the canonical map returns $\mu = 7.77$ at distance 144.5 and the linked map $\mu = 3.90$ at distance 42.6, both above the gap $[1.34, 2.0]$, the canonical one above every prior overshoot, and neither restores a peak. The two controls bracket the gap, pure radial at $\mu = 1.222$ and pure boundary at $\mu = 3.38$, yet no coupled map lands inside it, and neither improves on the standing-best distance 0.399. The coupling does not interpolate; the boundary multiplicity factor compounds the overshoot. The magnitude is shown to be in none of the radial, the boundary, or their coupling. No published number changes. The remaining direction is not a further arena but the framing itself, whether $(2, 3, 16)$ is an overlap-diagonal quantity, which the companion probe addresses.

Verifier. `verify_P320.py`: rebuilds the coupled operator $\hat{O}_{l,k}$ from scratch and confirms the two controls recover the two limits, the pure-radial control at $\mu = 1.222$ and the pure-boundary control at $\mu \geq 2.0$; confirms the full coupled spectrum $\lambda_{n,L} = \epsilon^{\text{rad}} + L(L + 2) + \text{Re}(\omega^k)\gamma$ is positive with the boundary Casimir $L(L + 2) = 0, 3, 8$ entering the spectrum; recomputes the canonical diagonal $(1, 7.77, 152.3)$ at distance 144.5 and the linked diagonal $(1, 3.90, 50.5)$ at distance 42.6, with no valid peak in either; confirms the coupled problem does not reach the gap $[1.34, 2.0]$, both μ above 2.0; confirms the controls bracket the gap while no coupled map lands inside it; and confirms the null verdict's defining condition, that the canonical map does not hit, the coupled problem does not reach the gap, and the coupled problem does not beat the standing-best distance 0.399, with no NaN. Nine checks, all PASS.