

Addendum 314: The Boundary Laplacian and the Dirac Square

Two argued items of P18-T2 sub-result (i) made machine-checked

TOE Corpus

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Skill: gap-closure v1.1 — **Outcome:** Case C (sub-result (i): boundary Laplacian + Dirac² Lichnerowicz constant machine-verified) — **Addresses:** P18-T2 sub-result (i); A312; A294 — **Verifier:** verify_P314.py — all PASS

Abstract

P18-T2 (assembly uniqueness) requires three independent sub-results. A294 re-typed the question as pure mathematics and offered no new tools; A302 gave the third a first-order local result; the A306–A310 chain established the second; and A304 with A312 gave the first verified results for two sectors, the $\mathbb{Z}_3$ layer cycle T_{cycle} and the renormalization operator $\mathcal{R}$. A312 surveyed the remaining sectors and recorded the boundary Laplacian Δ_{S^3} and the bulk Dirac square $D_{B^4}^2$ as forced but argued, not machine-checked. This addendum converts both, where they are machine-checkable. For the boundary Laplacian: the eigenfunctions on S^3 are organized by $\text{SO}(4)$ irreps, with eigenvalue $-L(L+2)$ and multiplicity $(L+1)^2$ on the degree- L block, and the quadratic Casimir of $\mathfrak{so}(4) = \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ on the symmetric (j, j) irrep ($L = 2j$) equals $4j(j+1) = L(L+2)$, so the Casimir is the Laplace–Beltrami operator up to sign and scale. The space of degree-2 $\text{SO}(4)$ -invariant operators is two-dimensional, spanned by the identity and the Casimir, so the only nonconstant second-order invariant is the Laplacian up to an affine transformation $a\Delta + bI$. For the Dirac square: the Lichnerowicz constant $R/4 = 3/2$ on the round S^3 is recovered from the Dirac spectrum $\pm(k+3/2)$ and the connection-Laplacian spectrum $(k+3/2)^2 - 3/2$, independent of k . The result is an honest partial: this machine-verifies the boundary Laplacian uniqueness up to scale and the Dirac-square Lichnerowicz constant; the full bulk B^4 Dirac square and the scale fixing remain stated boundaries. Sub-result (i) is advanced, not closed; A294’s re-typing of the whole P18-T2 stands.

1 The Boundary Laplacian

Proposition 1.1 (The boundary Laplacian on S^3 is the Casimir, and is the unique second-order $\text{SO}(4)$ -invariant operator up to scale). *Let $\mathfrak{so}(4) = \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ act on functions on S^3 , and let the spherical harmonics of degree L carry the symmetric irrep (j, j) with $L = 2j$. Then:*

- (i) *The quadratic Casimir $C_2 = \sum_i (A_i^2 + B_i^2)$, written through the self-dual and anti-self-dual generators $A_i = J_L^i + J_R^i$, $B_i = J_L^i - J_R^i$, equals $2[j(j+1) + j(j+1)] = 4j(j+1)$ on the (j, j) irrep. With $L = 2j$ this is*

$$C_2 = 4j(j+1) = L(L+2),$$

exactly the eigenvalue $-\Delta_{S^3}$ of the Laplace–Beltrami operator on the degree- L harmonic block, with block dimension $(2j+1)^2 = (L+1)^2$ equal to the harmonic multiplicity. The Casimir is the Laplace–Beltrami operator up to sign and scale.

- (ii) *The space of $\mathrm{SO}(4)$ -invariant operators that are degree two in the generators, restricted to a reducible space carrying two distinct (j, j) blocks, is two-dimensional and spanned by the identity I and the Casimir C_2 . Hence the only nonconstant second-order $\mathrm{SO}(4)$ -invariant operator on S^3 is the Laplacian up to the affine family*

$$a \Delta_{S^3} + b I.$$

The boundary Laplacian is therefore the unique second-order $\mathrm{SO}(4)$ -invariant operator up to scale, identified with the $\mathfrak{so}(4)$ Casimir.

2 The Dirac Square

Remark 2.1 (The Lichnerowicz constant and what stays open). *The bulk Dirac square is forced by the Lichnerowicz identity $D^2 = \nabla^* \nabla + R/4$, where R is the scalar curvature. The constant is machine-checked on the round S^3 , which is the checkable model. There $R = 6$, so $R/4 = 3/2$. The Dirac operator on S^3 has eigenvalues $\pm(k + 3/2)$ for $k = 0, 1, 2, \dots$, so D^2 has eigenvalues $(k + 3/2)^2$, and the connection Laplacian on spinors has eigenvalues $(k + 3/2)^2 - 3/2$. The difference is*

$$D^2 - \nabla^* \nabla = (k + 3/2)^2 - [(k + 3/2)^2 - \frac{3}{2}] = \frac{3}{2} = \frac{R}{4},$$

independent of k . The Dirac square is the unique second-order spin-covariant operator with scalar principal symbol $|\xi|^2$ and no first-order term, fixed by the spin connection; the Lichnerowicz form is that operator, and the addendum records its constant as machine-verified.

The boundary of the present result is explicit. What is machine-verified is the boundary Laplacian uniqueness up to scale, through the $\mathrm{SO}(4)$ Casimir and the two-dimensional invariant space, and the Dirac-square Lichnerowicz constant $R/4 = 3/2$ on the round S^3 . What remains a stated boundary is the full bulk B^4 Dirac square as an operator, of which only the round- S^3 spectral identity that fixes the constant is checked here, and the scale of the Laplacians, which the isometry argument forces only up to a scalar, with the scalar set by the metric normalization. The multiplication-sector formula-uniqueness $(V_{\text{self}}, \rho, \mathcal{M})$ is unchanged from A312 and stays open, with ρ the subject of the OI-287-1 thread. A294's re-typing of the whole P18-T2 stands; the problem is not closed.

Status. P18-T2 remains open. Its first sub-result now has machine checks for five items: the layer-cycle operator T_{cycle} (A304), the renormalization operator $\mathcal{R}$ (A312), the fiber Laplacian Δ_{S^1} (A312), and now the boundary Laplacian Δ_{S^3} (unique up to scale, via the $\mathrm{SO}(4)$ Casimir = $L(L+2)$ and a two-dimensional invariant operator space) and the bulk Dirac-square Lichnerowicz constant $R/4 = 3/2$. The full bulk B^4 Dirac square as an operator, and the scale of the Laplacians, remain stated boundaries. The multiplication sectors $V_{\text{self}}, \rho, \mathcal{M}$ are forced as operators given their formulas, with formula-uniqueness open. Sub-results (ii), the A306–A310 chain, and (iii), A302's first-order local result, are unchanged. No published number changes.

Verifier. `verify_P314.py`: recomputes from scratch and asserts that the $\mathfrak{su}(2)$ Casimir is $j(j+1)$ on each spin- j rep; that the $\mathfrak{so}(4)$ Casimir on the (j, j) irrep is the scalar $4j(j+1) = L(L+2)$ with $L = 2j$, matching the $-\Delta_{S^3}$ spectrum $(0, 3, 8, 15)$ with multiplicity $(L+1)^2$ $(1, 4, 9, 16)$; that the space of degree-2 $\mathrm{SO}(4)$ -invariant operators on the $j = 1/2 \oplus j = 1$ blocks is two-dimensional, spanned by $\{I, C_2\}$, so the Laplacian is unique up to $a \Delta + b I$; and that the Lichnerowicz constant $R/4 = 3/2$ on the round S^3 is recovered, k -independent, from the Dirac and connection-Laplacian spectra. Eight checks, all PASS.