

Addendum 304: The Layer-Cycle Operator is Forced

A verified result on P18-T2 sub-result (i) for one sector

TOE Corpus

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Skill: gap-closure v1.1 — **Outcome:** Case C (partial: one sub-result advanced for one sector)
 — **Addresses:** P18-T2 sub-result (i); A294 — **Verifier:** `verify_P304.py` — all PASS

Abstract

P18-T2 (assembly uniqueness) requires three independent sub-results; A294 re-typed the question as pure mathematics and offered no new tools, and A302 gave the third sub-result a first-order local result. This addendum advances the first sub-result, that each sector's named operator is the only operator in its category, for one sector where the forcing is demonstrable rather than asserted: the $\mathbb{Z}_3$ layer-cycle sector γT_{cycle} of P18 §3.4. The argument P18 sketches in Proposition (canonical selection) is made machine-checkable. The $\mathbb{Z}_3$ action, built as its regular representation, has exactly two nontrivial generators: the forward fiber rotation $\varphi \mapsto \varphi + 2\pi/3$, realized by the cyclic shift C carrying eigenvalue $\omega = e^{2\pi i/3}$ on the primitive character, and its inverse $C^2 = C^{-1}$, the backward rotation carrying $\omega^2 = \bar{\omega}$. Both satisfy $X^3 = I$ with spectrum the cube roots of unity, supplying the three-family assignment $k = 0, 1, 2$. The two generators are inverse, hence complex-conjugate, representations: $CC^2 = I$ and $C^2 = C^\top$. A single signed orientation invariant, the imaginary part of the eigenvalue carried on the fixed $k = 1$ Fourier character, takes value $+\sin(2\pi/3)$ on C and $-\sin(2\pi/3)$ on C^2 , and the two are exchanged by orientation reversal. Their quotients are the lens spaces $L(3, 1)$ and $L(3, 2) \cong -L(3, 1)$ (Reidemeister–Franz). Paper 06 requires $L(3, 1)$ and Paper 28 fixes the positive orientation, so the orientation invariant selects $C = T_{\text{cycle}}$ uniquely and excludes T_{cycle}^2 . For this sector the categorical operator is forced. The result is an honest partial: it addresses sub-result (i) for the layer-cycle sector only; the remaining sectors are untouched and A294's re-typing of the whole P18-T2 stands.

1 The Forcing

Proposition 1.1 (The layer-cycle operator is the unique categorical generator, given the fixed orientation). *Realize the $\mathbb{Z}_3$ layer-cycle action of P18 §3.4 as its regular representation, with the cyclic shift*

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C^2 = C^{-1} = C^\top,$$

cycling bulk $\rightarrow$ boundary $\rightarrow$ fiber $\rightarrow$ bulk. Then:

- (i) $\mathbb{Z}_3$ has exactly two nontrivial generators, C and C^2 , both of order three: $C^3 = (C^2)^3 = I$.
- (ii) The spectrum of either generator is the set of cube roots of unity $\{1, \omega, \omega^2\}$, $\omega = e^{2\pi i/3}$, carrying the three-family eigenspace assignment $k = 0, 1, 2$ of P18.

- (iii) C and C^2 are inverse, hence complex-conjugate, representations: $CC^2 = I$ and $C^2 = C^\top$; on the fixed primitive character $v_1 = (1, \omega, \omega^2)/\sqrt{3}$ the generator C carries ω and C^2 carries $\bar{\omega}$.
- (iv) The signed orientation invariant $\iota(X) = \text{Im}\langle v_1, Xv_1 \rangle$ satisfies $\iota(C) = +\sin(2\pi/3)$ and $\iota(C^2) = -\sin(2\pi/3)$; orientation reversal (complex conjugation) exchanges the two, matching $L(3, 2) \cong -L(3, 1)$ under Reidemeister–Franz.

Paper 06 §3 requires the positively oriented quotient $L(3, 1)$ and Paper 28 §3.2 fixes the positive orientation on $S^3 \subset \mathbb{C}^2$. The condition $\iota(X) > 0$ therefore selects exactly one generator, $C = T_{\text{cycle}}$, and excludes $C^2 = T_{\text{cycle}}^2$. The named operator of the layer-cycle sector is forced.

Remark 1.2 (What is forced, what remains open). The forcing is exact and finite: it rests on the order-three structure of $\mathbb{Z}_3$, the cube-root spectrum, the conjugacy of the two generators, and a single orientation invariant whose sign is fixed by two prior papers. No coefficient is tuned and no spectrum is fitted. The scope is narrow and stated as such. The result establishes per-category operator uniqueness for the layer-cycle sector alone, conditional on the orientation Papers 06 and 28 fix; without that orientation the two generators are interchangeable. The remaining sectors named in P18 §3 are not closed here. The kinetic operator $D_{B^4}^2$, the Laplacians Δ_{S^3} and Δ_{S^1} , the self-lensing potential, the density coupling ρ , and the moment hierarchy each require a separate categorical-uniqueness argument that this addendum does not supply. The renormalization generator $\mathcal{R} = r \partial_r + \Delta_\psi$ was examined and is reported as partial: the dilation generator $r \partial_r$ is the unique scale-transformation generator on the radial half-line up to scale, with eigenfunctions r^p and eigenvalue the scaling dimension, but the additive Δ_ψ is a c -number representation choice rather than an independent operator, so strict categorical uniqueness for that sector is not established. Sub-results (ii) additive composition and (iii) coefficient uniqueness are unchanged; (iii) carries A302’s first-order local result and (ii) is untouched. A294’s re-typing of the whole P18-T2 stands.

Status. P18-T2 remains open. Its first sub-result now has a verified result for one sector: the layer-cycle operator T_{cycle} is the unique categorical generator of the $\mathbb{Z}_3$ family sector, given the orientation Papers 06 and 28 fix, with T_{cycle}^2 excluded as the opposite-orientation quotient. The remaining sectors of sub-result (i) are open. Sub-results (ii) and (iii) are unchanged. No published number changes.

Verifier. `verify_P304.py`: builds the $\mathbb{Z}_3$ regular representation from scratch, asserts exactly two order-three generators with $C^3 = (C^2)^3 = I$, asserts the cube-root spectrum and the three-family assignment, asserts the conjugacy $CC^2 = I$ and $C^2 = C^\top$, asserts the orientation invariant flips sign between the two generators, and asserts the fixed positive orientation selects exactly one. Eight checks, all PASS.