

Addendum 301: The Family-Count Bound

The selection rule permits exactly three families

TOE Corpus

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Skill: gap-closure v1.1 — **Outcome:** Case A (a standing claim, P20's no-fourth-family, reproduced from an independent direction) — **Addresses:** OI-287-1 (boundary), P20 — **Verifier:** verify_P301.py — all PASS

Abstract

A297–A300 pursued the magnitude content of the selection rule: the factor that would lift the μ entry of $\langle \psi_n | c_k \pi^k x^k | \psi_n \rangle / \lambda_n$ from 1.12 to 1.5. That factor is still open. This addendum settles a separate, boundary question with the same machinery: how many families does the index-matching rule permit? The density is fixed by $\rho'(x) = 48\pi^3 x^2 + 6\pi^2 x + 2\pi$, hence $\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x$, a cubic. Its only nonzero Taylor coefficients are $(2, 3, 16)$ in units π^k at degrees 1, 2, 3; the degree-4 coefficient is identically zero. Family n draws the degree- n coefficient, so families 1, 2, 3 draw the nonzero $(2, 3, 16)$ while family 4 draws zero. The fourth eigenstate exists as an operator mode (node count 3, $\lambda_4 = 171.43$), but the coefficient it would couple to is absent, so it is decoupled. The selection rule therefore bounds the family count at $\deg \rho = 3$ with no additional postulate. This is the coupling-side image of P20's topological no-fourth-family, where the count is fixed by the three eigenspaces of $\mathbb{Z}_3 = \pi_1(S^3/\mathbb{Z}_3)$.

1 The Bound

Proposition 1.1 (The index-matching rule permits exactly three families). *Let $\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x$ be the density obtained by integrating $\rho'(x) = 48\pi^3 x^2 + 6\pi^2 x + 2\pi$, and let c_k denote its degree- k coefficient in units π^k : $c_1 = 2$, $c_2 = 3$, $c_3 = 16$, $c_4 = 0$. Under the index-matching rule, the diagonal coupling of family n is $\langle \psi_n | c_n \pi^n x^n | \psi_n \rangle / \lambda_n$. For the four lowest eigenstates of the production operator, which carry node counts 0, 1, 2, 3, the normalized diagonal is $(1, 1.12, 7.50, 0)$: the first three reproduce the standing lead, and the fourth is exactly zero because $c_4 = 0$. The fourth eigenstate is present and well-ordered; only its target coefficient is absent. The family count permitted by the rule equals the number of nonzero coefficients of ρ , which is $\deg \rho = 3$.*

Remark 1.2 (Two readings of the same three). *The bound is not an independent derivation of the number three. It is a demonstration that two pieces of the framework agree on it. P20 fixes three families topologically, as the eigenspaces of the layer-cycle operator on $S^3/\mathbb{Z}_3$; the present argument fixes the same three arithmetically, as the nonzero coefficients of a cubic density. The content is the consistency: the same selection rule that gives the right $(2, 3, 16)$ direction for $n = 1, 2, 3$ also forbids $n = 4$ when applied one step further, without a separate prohibition. The deeper question, why ρ is cubic, is the same question as why $\mathbb{Z}_3$ and not $\mathbb{Z}_m$, and is not answered here. What is answered is that the lepton-coupling sector and the topological sector do not disagree about the family count.*

Status. OI-287-1 remains open in its magnitude part: the localizing factor that lifts the μ entry and concentrates the coupling on the diagonal (A300). Its boundary part is settled: the rule permits exactly $\deg \rho = 3$ families, in agreement with P20. P20's claim that a fourth generation is forbidden gains an independent witness. No published number changes.

Verifier. `verify_P301.py`: confirms ρ is cubic with coefficients $(2, 3, 16)$ and $c_4 = 0$, confirms a fourth eigenstate exists with node count 3, and confirms families 1–3 keep the $(1, 1.12, 7.50)$ lead while family 4 draws zero. All PASS.