

Addendum 276: The Closure-Map Family

Split idempotents, defect functors,
and the quantification of “dark” (OI-272-1 closed)

TOE Corpus

2026-06-10

Skill: gap-closure v1.1 — **Outcome:** Case A (formalization; family laws proven; coverage claim quantified) — **Closes:** OI-272-1 — **Verifier:** `verify_P276.py` — all PASS

Abstract

A272 defined dark as the dynamics of the kernel of a “closure map of the corpus’s collapse–projection family,” quantifying over a list (fold, chord closure, locking) rather than a class; OI-272-1 asked for the class. It is supplied here. A *closure datum* is a retraction pair $(P: X \rightarrow X_0, C: X_0 \rightarrow X)$ with $PC = \text{id}$, equivalently a split idempotent $e = CP$ on X ; this is the formal content of the corpus’s $C \circ P = I$ (P27, P34), read as an identity on the canonical sector rather than on all of X (on all of X it would force invertibility — the precise objection of P27’s Open remark P027.2 on the 2-to-1 Hopf map, which this typing repairs without retracting). *Dark* is the defect of e from id_X , measured by a defect functor fixed per ambient category: kernel (linear ambient Hilb), residue distance $x \mapsto d(x, e(x))$ (metric ambient), divergence locus of the locking projection (dynamical ambient). The three A272 instances are members (Proposition 1.3); A254’s collapse delta $\delta_\Omega = \text{rank ker}(H_0(A) \rightarrow H_0(X))$ is the defect functor of the topological ambient, confirming A272’s prototype claim. *Family laws* (Theorem 2.1): closure data compose — (P_2P_1, C_1C_2) is again a closure datum — and in the linear ambient the kernel ladder $0 \rightarrow \text{ker } P_1 \rightarrow \text{ker}(P_2P_1) \rightarrow \text{ker } P_2$ is exact, so *dark accumulates along composition*: deeper closure means larger kernel. A272’s consonance remark (dark fraction $\ell/(\ell+1) \rightarrow 1$ at depth) is thereby upgraded from consonance to structure: kernel growth under composed closure is a theorem of the family, not a property of one instance. The coverage claim is restated over the defined class and remains falsifiable. Wheel heat (A261’s thermalized defect) is typed as the quantitative defect — the energy carried by $\text{id} - e$ — consistent with A272’s watch item; flagged, not developed.

1 The Family

Definition 1.1 (Closure datum; the family $\mathcal{F}$). *Let $\mathcal{A}$ be one of the corpus’s ambient categories: Hilb (mode spaces), Met (metric state spaces with distinguished lattices), Dyn (circle/sphere maps). A closure datum in $\mathcal{A}$ is a pair $(P: X \rightarrow X_0, C: X_0 \rightarrow X)$ with $PC = \text{id}_{X_0}$. Its closure operator is the idempotent $e = CP: X \rightarrow X$ ($e^2 = C(PC)P = CP = e$); its canonical sector is $\text{im } e \cong X_0$. The family $\mathcal{F}$ is the class of all closure data over the three ambients. The corpus’s $C \circ P = I$ is the statement $e|_{\text{im } e} = \text{id}$ — exact on the canonical sector, idempotent (not invertible) globally.*

Definition 1.2 (Defect functors; dark). *Per ambient, the defect of a closure datum is: Hilb: $\text{ker } P$ (equivalently $\text{im}(\text{id} - e)$); Met: the residue function $x \mapsto d(x, e(x))$ (commas); Dyn: the locus where the locking projection fails to converge (drift). Dark (A272 Def. 1.1) is the dynamics of the defect. In the topological ambient the defect is A254’s collapse delta $\delta_\Omega(A, X) = \text{rank ker}(H_0(A) \rightarrow H_0(X))$, which is hereby the defect functor of Top, confirming A272’s prototype identification.*

Proposition 1.3 (The three instances are members). (I1) Fold (Hilb): $P = \text{Hopf pushforward on mode spaces}$, $C = \text{inclusion of the charge-blind sector}$; $PC = \text{id}$ because the pushforward reproduces the charge-blind radial density exactly (A269); $\ker P = \text{the charged sector}$, rank $\ell(\ell + 1)$ per level (A259). (I2) Chord closure (Met): $X = \mathbb{R}_{>0}$ (cycle lengths), $X_0 = \text{the 3-smooth lattice}$, $C = \text{inclusion}$, $P = \text{nearest-chord projection}$; $PC = \text{id}$ (a chord is its own nearest chord); residue at T_{breath} is G_1 (A266). (I3) Locking (Dyn): $X_0 = \text{locked orbits (tongues)}$, $P = \text{the phase-locking projection defined on tongue basins}$, $C = \text{inclusion}$; defect locus = the drifting class, non-empty since ρ is irrational (A267/A268).

Remark 1.4 (P27’s Open remark P027.2, typed without retraction). P27 (Open remark P027.2) observes that the Hopf map is 2-to-1, so $C \circ P = I$ cannot hold by invertibility, and records the gap as open. Definition 1.1 is the repair: the identity was never global — it is the section law $PC = \text{id}_{X_0}$ plus idempotency, which a 2-to-1 (indeed any split) projection supports without contradiction. The non-injectivity is not an obstruction; it is exactly where dark lives ($\ker P \neq 0$). The remark’s demand for invertibility dissolves; the closure is recorded in the A277 registry.

2 Family Laws

Theorem 2.1 (Composition; the kernel ladder). (i) If (P_1, C_1) on X with canonical X_0 and (P_2, C_2) on X_0 with canonical X_{00} are closure data, then (P_2P_1, C_1C_2) is a closure datum on X : $P_2P_1C_1C_2 = P_2C_2 = \text{id}$. $\mathcal{F}$ is closed under composition (and contains identities), hence forms a category fibred over the ambients. (ii) In Hilb the sequence

$$0 \rightarrow \ker P_1 \rightarrow \ker(P_2P_1) \xrightarrow{P_1} \ker P_2 \rightarrow 0 \quad (1)$$

is exact (the right map is onto because C_1 sections it: $P_1C_1 = \text{id}$ maps $\ker P_2$ into $\ker(P_2P_1)$), so $\dim \ker(P_2P_1) = \dim \ker P_1 + \dim \ker P_2$: dark accumulates additively along composed closure.

Corollary 2.2 (Dark dominance at depth is structural). Iterating closure grows the defect monotonically; no composite of closure data can shrink a kernel. A272’s consonance remark — per-level dark fraction $\ell/(\ell + 1) \rightarrow 1$, “kernel dominance at depth” — is an instance of the ladder, not a peculiarity of the fold. A universe reached through many closures is generically dark-dominated.

Remark 2.3 (Wheel heat as quantitative defect — flag). A261’s thermalized defect (Wheel heat) is naturally the energy of $\text{id} - e$: the quantitative size of the defect rather than its support. This types A272’s watch item consistently (heat = kernel energy after support removal) but is recorded as a flag — the energy form on $\text{id} - e$ is not developed here.

Coverage, restated over the class. A272’s falsifiable claim now reads: every corpus “dark” phenomenon is the defect of a closure datum in $\mathcal{F}$ (Definition 1.1), under one of the three defect functors (Definition 1.2). A dark phenomenon admitting no such presentation breaks the claim. The i_Paper’s Nullity (kernel of frame installation, Hilb-like) and dark energy (cokernel — the dual defect, A272 Rem. 2.4) remain consistent; cokernels enter as the defect of the *opposite* datum $(C^\dagger, P^\dagger)$ in Hilb, which is again in $\mathcal{F}$.

Consequences. OI-272-1 closed; A272’s definition quantifies over a defined class. The kernel ladder gives the corpus a new structural slogan with theorem backing: closure composes, dark adds.

Verifier. `verify_P276.py`: idempotency and section laws on explicit Hilb instances (random splittings, 50-digit); composition law; exactness of the kernel ladder (rank arithmetic on explicit matrices); I1 rank recomputation; I2 residue = G_1 and idempotency of nearest-chord on a 3-smooth grid; I3 non-empty defect (irrationality, tongue distance); cokernel-as-opposite-datum check. All PASS.