

DRAFT Addendum 255: Two Primitive Threes and the Derived Hopf–Lens Bridge

Quaternionic Hopf Closure, Lens-Space Families,
and the Independence of Triality

L. F. Vlegels
Independent Researcher
axiomofchoicepuzzle@gmail.com

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Abstract

This addendum resolves the TASK 022 bridge question. The quaternionic Hopf side closes: P01’s Cayley–Dickson split $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$ supplies an $\mathbb{H}^2$ coordinate model, and the projection $S^7_{\mathbb{H}^2} \rightarrow \mathbb{H}P^1 \simeq S^4$ has S^3 fibers. The fiber over $[1 : 0]$ is exactly the embedded unit-quaternion $S^3 \subset S^7$ already present in P01.

The Hopf-fiber-to-lens step also closes. On a unit quaternion $q = z_1 + z_2j$, left multiplication by $\epsilon = e^{2\pi i/3}$ acts as $(z_1, z_2) \mapsto (\epsilon z_1, \epsilon z_2)$, exactly the P01 generator of $L(3, 1) = S^3/\mathbb{Z}_3$.

The stronger proposed identification does not close. The triality subgroup $C_3 < \text{Out}(\text{Spin}(8))$ and the lens deck group $\mathbb{Z}_3 < \text{Diff}(S^3)$ are provably independent as typed objects. The lens generator moves $1 \in S^3$ to ϵ , while every element of $G_2 = \text{Aut}(\mathbb{O})$ fixes 1 and the real scalars. Therefore the lens generator is not in G_2 and cannot be obtained through $Z(SU(3)) \subset SU(3) \subset G_2$. The canon must state two primitive threes: the lens-space topological $\mathbb{Z}_3$ and the representation-theoretic triality C_3 . A44’s cyclic permutation of the three off-diagonal entries of $J_3(\mathbb{O})$ is a third, distinct order-three object.

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1 Typed Objects

Definition 1.1 (Cayley–Dickson coordinate model). P01 defines the octonions by the Cayley–Dickson split

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$$

at 01_Paper_PerfectStableSphere.tex:805–812. Define

$$\Phi : \mathbb{H}^2 \rightarrow \mathbb{O}, \quad \Phi(a, b) = a + b\ell.$$

Let

$$S_{\mathbb{H}^2}^7 = \{(a, b) \in \mathbb{H}^2 : \|a\|^2 + \|b\|^2 = 1\}.$$

Remark 1.2 (Module action, not octonion multiplication). The right $\mathbb{H}$ action used for the Hopf fibration is the module action

$$(a + b\ell) \cdot_R h := ah + (bh)\ell.$$

It is not literal octonion multiplication by h . Under octonion multiplication the second component is conjugated in the Cayley–Dickson rule; for example $a = 0$, $b = 1$, $h = i$ gives $(0, i)$ for the module action and $(0, -i)$ for the octonion product. This typing prevents a false associativity claim.

2 Joint A: Quaternionic Hopf Closure

Theorem 2.1 (Quaternionic Hopf closure). *After choosing the P01 Cayley–Dickson split, the unit octonion sphere is the total space of the quaternionic Hopf fibration*

$$S^3 \longrightarrow S_{\mathbb{H}^2}^7 \longrightarrow \mathbb{H}P^1 \simeq S^4.$$

Proof. The Cayley–Dickson norm gives

$$\|a + b\ell\|^2 = \|a\|^2 + \|b\|^2,$$

so Φ identifies $S_{\mathbb{H}^2}^7$ with $S_{\mathbb{O}}^7 = \{x \in \mathbb{O} : \|x\| = 1\}$. Define

$$\pi_{\mathbb{H}}(a, b) = [a : b] \in \mathbb{H}P^1.$$

The right unit-quaternion action

$$(a, b) \cdot u = (au, bu), \quad \|u\| = 1,$$

has orbits equal to the fibers of $\pi_{\mathbb{H}}$, so each fiber is S^3 . The base $\mathbb{H}P^1$ is diffeomorphic to S^4 . □

Corollary 2.2 (P01 embedded sphere is a Hopf fiber). *The fiber over $[1 : 0]$ is*

$$\{(u, 0) : \|u\| = 1\} \cong S^3.$$

Under Φ , this is exactly P01’s embedded unit-quaternion sphere $S^3 \subset S^7$ from 01_Paper_PerfectStable

3 The Lens Generator on the Hopf Fiber

Theorem 3.1 (Fiber-to-lens closure). *Let $q = z_1 + z_2j$ be a unit quaternion, with $z_1, z_2 \in \mathbb{C}$. Let $\epsilon = e^{2\pi i/3} \in \mathbb{C} \subset \mathbb{H}$. Left multiplication by ϵ acts on the Hopf fiber S^3 by*

$$q \mapsto \epsilon q \iff (z_1, z_2) \mapsto (\epsilon z_1, \epsilon z_2).$$

This is the P01 generator of $L(3, 1) = S^3/\mathbb{Z}_3$.

Proof. Since $\epsilon \in \mathbb{C} \subset \mathbb{H}$ acts on the left,

$$\epsilon(z_1 + z_2j) = \epsilon z_1 + \epsilon z_2j.$$

P01 records the $p = 3, q = 1$ lens generator as $(z_1, z_2) \mapsto (\omega z_1, \omega z_2)$ at `01_Paper_PerfectStableSphere`.

Taking $\omega = \epsilon$ gives the same action. The action is free: if $\epsilon q = q$, then $(\epsilon - 1)q = 0$; since $\mathbb{H}$ is a division algebra and $\epsilon \neq 1$, this forces $q = 0$, impossible on S^3 . $\square$

4 Triality Is Independent

Proposition 4.1 (The lens generator is not in G_2). *The lens generator $L_\epsilon : S^3 \rightarrow S^3$, $q \mapsto \epsilon q$, is not the restriction of any element of $G_2 = \text{Aut}(\mathbb{O})$.*

Proof. The point $1 \in S^3 \subset \mathbb{H}$ is sent by the lens generator to $\epsilon \neq 1$. But $G_2 = \text{Aut}(\mathbb{O})$ fixes the real unit. P01 defines $G_2 = \text{Aut}(\mathbb{O})$ at `01_Paper_PerfectStableSphere.tex:814-818`; A44 explicitly records that $G_2 = \text{Aut}(\mathbb{O})$ fixes $e_0 = 1$ and the real subspace at `44_Addendum_G2ActionJ3`. Hence no $g \in G_2$ can agree with L_ϵ on S^3 . $\square$

Corollary 4.2 (No $SU(3)$ -center route). *The lens deck transformation is not obtained through $Z(SU(3)) \subset SU(3) \subset G_2$.*

Proof. Every element of $SU(3)$ in the corpus route is an element of G_2 , and so fixes 1. The lens generator does not fix 1. $\square$

Proposition 4.3 (Triality and lens Z_3 are independent). *The cyclic triality subgroup $C_3 < \text{Out}(\text{Spin}(8))$ and the lens deck group $\mathbb{Z}_3 < \text{Diff}(S^3)$ are not the same structure in the current canon.*

Proof. Triality is used in P01/P05 as a representation-theoretic order-three structure supporting the bulk coefficient 16; see `01_Paper_PerfectStableSphere.tex:371-374` and `05_Paper_GeometryOfReality.tex:590-596`. The lens group acts on points of the Hopf fiber S^3 . No corpus-defined map

$$C_3 < \text{Out}(\text{Spin}(8)) \longrightarrow \text{Diff}(S^3)$$

sends the triality generator to L_ϵ , and the $G_2/SU(3)$ route is blocked by the preceding proposition. $\square$

5 The Third Order-Three Structure

A44 contains a separate $\mathbb{Z}_3$ acting on the off-diagonal slots of $J_3(\mathbb{O})$:

$$\sigma : (x_{12}, x_{13}, x_{23}) \mapsto (x_{23}, x_{12}, x_{13}).$$

This is recorded at `44_Addendum_G2ActionJ3.tex:396-404` and summarized at `44_Addendum_G2ActionJ3.tex:405-410`. It acts on Jordan-algebra slots, not on the Hopf fiber S^3 , and it is not the triality subgroup used for the P01/P05 bulk coefficient.

6 Canonical Axioms

Axiom 6.1 (LS: lens-space topology). The boundary Hopf fiber S^3 carries the free order-three action generated by left multiplication by $\epsilon = e^{2\pi i/3}$. The quotient is $L(3, 1) = S^3/\mathbb{Z}_3$, and this is the topological source of the three fermion-family covering.

Axiom 6.2 (TR: triality). The octonionic bulk carries the representation-theoretic order-three triality of $\text{Spin}(8)$, permuting the vector and two half-spinor representations. In the current corpus this supports the bulk coefficient $16 = 8 + 8$.

Remark 6.3 (Do not merge the threes). The canon may relate LS, TR, and the A44 $J_3(\mathbb{O})$ permutation structurally, but it may not identify them without an explicit typed map. The $G_2/SU(3)$ route cannot supply that map for the lens generator because G_2 fixes $1 \in \mathbb{O}$ while the lens generator moves it.

7 Status Table

Claim	Status	Canon action
Quaternionic Hopf from P01 split	DERIVED	Add to A255 as theorem.
P01 embedded S^3 is Hopf fiber	DERIVED	Add as corollary.
Fiber C_3 gives $L(3, 1)$	DERIVED	Add as theorem.
Triality C_3 equals lens $\mathbb{Z}_3$	NEGATIVE	State independence.
A44 $J_3(\mathbb{O})$ permutation $\mathbb{Z}_3$	DISTINCT	Catalogue separately.