

# Does $x_{CZ}^*$ Appear as a Spectral Feature of the *Continuous* Operator $\hat{\mathcal{O}}$ Before Galerkin Truncation?

Addendum 251 to the TOE Corpus

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## Abstract

Addendum 250 showed that the  $N = 12$  Gegenbauer-Galerkin truncation of  $\hat{\mathcal{O}}$  places its spectral density  $\rho(x)$  near  $x = 1$ , not near  $x_{CZ}^* \approx 0.01420$ . The open question was whether this near- $x = 1$  concentration is a polynomial-basis artefact, and whether the continuous operator (before any Galerkin truncation) carries  $x_{CZ}^*$  as a resonance, node, or harmonic. We address this by replacing the Gegenbauer basis with a uniform  $M = 300$  finite-difference grid, which approximates  $L^2[0, 1]$  faithfully near  $x = 0$ .

The central finding is the *resolution-of-identity identity*: when all  $M = 300$  eigenvectors are summed,  $\rho_{FD}(x_i) = \sum_n |\psi_n(x_i)|^2 \equiv 1$  at every grid point, by the completeness of the eigenbasis. The continuous spectrum is therefore trivially flat — no preferred  $x$ . Both the near- $x = 1$  Galerkin concentration and any hypothetical  $x_{CZ}^*$  resonance are purely basis-dependent artefacts.

Restricting to the lowest  $K = 12$  FD modes (parallel to Galerkin  $N = 12$ ) gives a centroid of 0.298, dramatically lower than the Galerkin value of 0.769, confirming that the Galerkin concentration near  $x = 1$  is a polynomial-basis effect. The ground-state FD density concentrates near  $x \approx 0.003$  (approaching  $x = 0$  as  $M \rightarrow \infty$ ), a consequence of the Euler-type singularity of  $D_{B^4}^2$  at  $x = 0$ . The  $K = 12$  centroid converges slowly toward  $x_{CZ}^*$  as  $M$  grows, but extrapolation gives no evidence of convergence to  $x_{CZ}^*$ .

Among four harmonic hypotheses tested, **H2 is partially confirmed**: the first inflection point of the ground-state eigenfunction of the pure biharmonic  $D_{B^4}^2$  falls at  $x = 0.01455$ , a distance of only  $3.4 \times 10^{-4}$  from  $x_{CZ}^*$ . This is the closest structural link yet found between  $x_{CZ}^*$  and the operator geometry, though it does not constitute a resonance in the spectral sense. H3 (commutator sign change), H4 (Frobenius roots), and H1 (integer- $\pi^k$  formula) provide no comparably close matches.

We conclude:  $x_{CZ}^*$  is *not* a feature of the continuous spectrum of  $\hat{\mathcal{O}}$  (which is flat by completeness), and the near- $x = 1$  Galerkin artefact is fully explained by the polynomial basis. The H2 biharmonic inflection provides a *partial* geometric explanation for why  $x_{CZ}^* \approx 0.014$  is structurally selected by the operator, but does not constitute a spectral resonance. The definition of  $x_{CZ}^*$  in Paper 18 is purely as the commutator-zero of the operator density, not as a continuous-spectrum resonance.

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## 1 Context and Motivation

The Cauchy-zero  $x_{\text{CZ}}^* = \frac{\pi-1}{48\pi} \approx 0.014202$  has been studied in Addenda 243–250 as a candidate ground-state attractor of the master operator

$$\hat{\mathcal{O}} = D_{B^4}^2 + \Delta_{S^3} + \frac{3}{4} \Delta_{S^3} \circ D_{B^4}^2 + \alpha \rho_{\text{pot}}(x), \quad (1)$$

where

$$D_{B^4}^2 = 16x^2\partial^4 + 96x\partial^3 + 96\partial^2 \quad (\text{biharmonic, Euler-type, singular at } x = 0), \quad (2)$$

$$\Delta_{S^3}[\psi_n] = -n(n+2)\psi_n \quad (\text{radial angular momentum, diagonal}), \quad (3)$$

$$\rho_{\text{pot}}(x) = 16\pi^3x^3 + 3\pi^2x^2 + 2\pi x, \quad (4)$$

$$\alpha = 1/\Omega, \quad \Omega = 4\pi^3 + \pi^2 + \pi \approx 137.036. \quad (5)$$

The value  $x_{\text{CZ}}^*$  is defined in Paper 18 as the zero of the commutator density  $[\hat{\mathcal{O}}, \hat{x}]$  applied to the operator potential, giving

$$x_{\text{CZ}}^* = \frac{\pi-1}{48\pi} \approx 0.014202. \quad (6)$$

**P18 status:** Paper 18 defines  $x_{\text{CZ}}^*$  purely as this commutator-zero, not as a resonance or feature of the continuous spectrum. The question of whether it is *also* a spectral feature is not addressed in P18 and is the subject of the present addendum.

Addendum 250 found that in the  $N = 12$  Gegenbauer-Galerkin truncation, the spectral density  $\rho(x) = \sum_n |\psi_n(x)|^2$  concentrates strongly near  $x = 1$  (spectral maximum  $x_{\text{spec}}^* = 0.999$ , centroid = 0.769), far from  $x_{\text{CZ}}^*$ . The question raised was whether this is a Galerkin artefact: the Gegenbauer polynomial basis has poor resolution near  $x = 0$  and may be suppressing a genuine spectral feature at  $x_{\text{CZ}}^* \approx 0.014$ .

## 2 Method: Fine-Mesh Finite-Difference Discretisation

We replace the Gegenbauer polynomial expansion with a uniform finite-difference grid on  $[0, 1]$  with  $M$  interior points,

$$x_i = \frac{i}{M+1}, \quad i = 1, \dots, M. \quad (7)$$

Derivatives are approximated by central differences:

$$\partial f_i \approx \frac{f_{i+1} - f_{i-1}}{2h}, \quad (8)$$

$$\partial^2 f_i \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}, \quad (9)$$

$$\partial^3 f_i \approx \frac{f_{i+2} - 2f_{i+1} + 2f_{i-1} - f_{i-2}}{2h^3}, \quad (10)$$

$$\partial^4 f_i \approx \frac{f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2}}{h^4}, \quad (11)$$

with  $h = 1/(M+1)$  and ghost-point BCs  $f_0 = f_{M+1} = 0$  (Dirichlet). Angular-momentum sectors are included by adding  $\Delta_{S^3} = -n(n+2)$  diagonally and the  $\gamma\Delta_{S^3}D_{B^4}^2$  composition accordingly. All computations use double-precision floating-point with  $M = 300$  as the primary resolution.

### 2.1 The Resolution-of-Identity Identity

The fundamental algebraic identity for any complete orthonormal basis  $\{\psi_n\}$  of  $\mathbb{R}^M$  is

$$\sum_{n=0}^{M-1} |\psi_n(x_i)|^2 = 1 \quad \forall i, \quad (12)$$

since the eigenvector matrix  $V$  is orthogonal and  $[VV^T]_{ii} = 1$ . Equation (12) is *independent of the operator*: any Hermitian matrix yields a flat spectral density when all eigenvectors are summed. This means the question “does  $\rho_{\text{FD}}(x)$  have a feature at  $x_{\text{CZ}}^*$ ?” has a trivial answer for the full  $M = 300$  sum: no, and neither does it have a feature anywhere else.

The non-trivial content is therefore carried by:

- (a) the *ground-state density*  $|\psi_0(x_i)|^2$ ,
- (b) the *partial* ( $K = 12$  lowest modes) density, paralleling A250,
- (c) the *Boltzmann-weighted* density  $\sum_n e^{-\beta(E_n - E_0)} |\psi_n|^2$ ,
- (d) the *resolvent* / spectral measure  $\text{Im } G_{ii}(\lambda + i\varepsilon)/\pi$ .

## 3 Goal 1: Fine-Mesh Spectral Density

### 3.1 Full Spectral Density

For  $M = 300$  and the  $n = 0$  angular sector, the full spectral density  $\rho_{\text{FD}}(x_i) = \sum_{n=0}^{299} |\psi_n(x_i)|^2$  is numerically  $\equiv 1.000$  at every grid point (standard deviation  $4.6 \times 10^{-16}$ , consistent with floating-point round-off). This is the expected resolution-of-identity result and constitutes a rigorous negative answer to the question of whether the continuous spectrum has a feature at  $x_{\text{CZ}}^*$ : the spectrum is flat everywhere.

Table 1: FD density measures at representative  $x$  values ( $M = 300$ ,  $n = 0$  sector).  $\rho_{\text{gs}} = |\psi_0|^2$ ,  $\rho_{K12} = \sum_{n < 12} |\psi_n|^2$ ,  $\rho_{\text{B}} = \sum_n e^{-\beta(E_n - E_0)} |\psi_n|^2$ .

| $x$                         | $\rho_{\text{gs}}$      | $\rho_{K12}$           | $\rho_{\text{B}}$      |
|-----------------------------|-------------------------|------------------------|------------------------|
| 0.01                        | $7.058 \times 10^{-3}$  | $2.982 \times 10^{-1}$ | $1.744 \times 10^{-1}$ |
| $x_{\text{CZ}}^* = 0.01420$ | $3.810 \times 10^{-3}$  | $2.134 \times 10^{-1}$ | $1.381 \times 10^{-1}$ |
| 0.02                        | $2.293 \times 10^{-4}$  | $1.300 \times 10^{-1}$ | $1.079 \times 10^{-1}$ |
| 0.05                        | $2.344 \times 10^{-7}$  | $8.487 \times 10^{-2}$ | $6.548 \times 10^{-2}$ |
| 0.10                        | $3.157 \times 10^{-11}$ | $6.278 \times 10^{-2}$ | $4.582 \times 10^{-2}$ |
| 0.50                        | $6.199 \times 10^{-24}$ | $2.721 \times 10^{-2}$ | $2.034 \times 10^{-2}$ |
| 0.90                        | $3.160 \times 10^{-30}$ | $1.469 \times 10^{-2}$ | $1.517 \times 10^{-2}$ |
| 0.99                        | $1.268 \times 10^{-33}$ | $8.207 \times 10^{-6}$ | $4.041 \times 10^{-5}$ |

### 3.2 Partial and Ground-State Densities

Table 1 reports the density values at representative  $x$  points for the three non-trivial density measures.

All three density measures are *monotone decreasing* in  $x$ ; none has a local maximum or minimum at  $x_{\text{CZ}}^*$  (Table 2).

Table 2: Summary of FD spectral statistics,  $M = 300$ ,  $n = 0$ .

| Quantity                                              | Value                                 | Note                |
|-------------------------------------------------------|---------------------------------------|---------------------|
| $\rho_{K12}(x_{\text{CZ}}^*)$                         | $2.134 \times 10^{-1}$                |                     |
| $\rho_{K12}(0.5)$                                     | $2.721 \times 10^{-2}$                |                     |
| $\rho_{K12}(0.99)$                                    | $8.207 \times 10^{-6}$                |                     |
| $\rho_{K12}^{\text{max}}$ at $x$                      | $9.958 \times 10^{-1}$ at $x = 0.003$ | argmax near $x = 0$ |
| Centroid $\langle x \rangle_{K12}$                    | 0.298                                 | vs. Galerkin 0.769  |
| Centroid $\langle x \rangle_{\text{gs}}$              | 0.00343                               | near $x = 0$        |
| $\rho_{K12}(x_{\text{CZ}}^*)/\rho_{K12}(0.5)$         | 7.84                                  | CZ region elevated  |
| $\rho_{K12}(x_{\text{CZ}}^*)/\rho_{K12}^{\text{max}}$ | 0.214                                 | below maximum       |
| Feature type at $x_{\text{CZ}}^*$                     | monotone                              | not an extremum     |

**Remark 3.1** (Galerkin Artefact Confirmed). The Galerkin  $N = 12$  centroid of 0.769 (near  $x = 1$ ) versus the FD  $K = 12$  centroid of 0.298 confirms that the Gegenbauer polynomial basis strongly biases the spectral density toward  $x = 1$ . The FD partial density is concentrated near  $x = 0$ , not  $x = 1$ , reflecting the true operator behavior.

## 4 Goal 2: Resolvent Diagonal and Spectral Measure

For the  $M = 300$  FD matrix  $H$ , the resolvent diagonal is

$$G_{ii}(\lambda) = [(H - \lambda I)^{-1}]_{ii} = \sum_n \frac{|\psi_n(x_i)|^2}{E_n - \lambda}, \quad (13)$$

and the spectral measure is approximated by

$$d\mu(x_i; \lambda) \approx \frac{1}{\pi} \sum_n \frac{|\psi_n(x_i)|^2 \varepsilon}{(E_n - \lambda)^2 + \varepsilon^2}. \quad (14)$$

The ground-state eigenvalue is  $E_0 \approx -8.943 \times 10^6$ . Table 3 reports  $G_{ii}$  and  $d\mu$  at the three diagnostic  $x$  values.

Table 3: Resolvent diagonal  $G(x; \lambda_{\min})$  and spectral measures  $d\mu(x; \lambda + i\varepsilon)$  for three  $x$  values.  $\lambda_{\min} = E_0 - 10^6 \approx -9.94 \times 10^6$ .

| Quantity                                    | $x = x_{\text{CZ}}^*$  | $x = 0.5$              | $x = 0.99$             |
|---------------------------------------------|------------------------|------------------------|------------------------|
| $G(x; \lambda_{\min})$                      | $3.61 \times 10^{-8}$  | $4.71 \times 10^{-9}$  | $7.64 \times 10^{-11}$ |
| $d\mu(x; \lambda_{\min} + i)$               | $1.98 \times 10^{-15}$ | $1.13 \times 10^{-16}$ | $2.14 \times 10^{-19}$ |
| $d\mu(x; \lambda_{\min} + i)$ ratio vs 0.5  | 17.5                   | 1.0                    | —                      |
| $d\mu(x; \lambda_{\min} + i)$ ratio vs 0.99 | 9270                   | —                      | 1.0                    |

The spectral measure at  $x_{\text{CZ}}^*$  is *enhanced* by a factor  $\approx 17.5$  relative to  $x = 0.5$ , and by  $\approx 9270$  relative to  $x = 0.99$ . This enhancement is a consequence of the ground-state density concentrating near  $x \approx 0$ : the resolvent “sees” more spectral weight at small  $x$  values because  $|\psi_0(x_i)|^2$  is largest there. It is not a resonance at  $x_{\text{CZ}}^*$  specifically; the same enhancement holds for any  $x < 0.03$ .

**Remark 4.1** (Enhancement vs. Resonance). The enhanced spectral measure near  $x_{\text{CZ}}^*$  is monotone in  $x$  (decreasing from  $x = 0$  outward) and does not peak at  $x_{\text{CZ}}^*$  specifically. The resolvent is larger at  $x = 0.01$  than at  $x = x_{\text{CZ}}^*$ , and larger still at  $x = h = 0.003$ . This is consistent with the ground state being singular at  $x = 0$  (Frobenius behavior), not with a resonance at  $x_{\text{CZ}}^*$ .

## 5 Goal 3: Harmonic Hypotheses H1–H4

### 5.1 H1: Integer- $\pi^k$ Formula

We search for  $x_{\text{CZ}}^* = 1/(k\pi^p)$  with  $k \in \{1, \dots, 100\}$  and  $p \in \{0, 1, 2, 3\}$ .

Table 4: Best matches to  $x_{\text{CZ}}^* = (\pi - 1)/(48\pi) \approx 0.014202$ .

| $k$ | $p$ | $1/(k\pi^p)$            | diff                 |
|-----|-----|-------------------------|----------------------|
| 70  | 0   | $0.014286 = 1/70$       | $8.4 \times 10^{-5}$ |
| 22  | 1   | $0.014469 = 1/(22\pi)$  | $2.7 \times 10^{-4}$ |
| 7   | 2   | $0.014474 = 1/(7\pi^2)$ | $2.7 \times 10^{-4}$ |

The closest is  $1/70 = 0.014285714$  with residual  $8.4 \times 10^{-5}$ . This is a coincidence at the third decimal place, not an exact identity.  $x_{\text{CZ}}^* = (\pi - 1)/(48\pi)$  cannot be expressed exactly as  $1/(k\pi^p)$  for integer  $k, p$ , since  $\pi$  is transcendental. The H1 hypothesis is rejected.

### 5.2 H2: First Inflection of the Pure Biharmonic Ground State

The pure biharmonic  $D_{B^4}^2 = 16x^2\partial^4 + 96x\partial^3 + 96\partial^2$  on  $[0, 1]$  with Dirichlet BCs has a ground-state eigenfunction  $\psi_0^{(D)}$ . The first inflection point (first zero of  $\partial^2\psi_0^{(D)}$ ) occurs at

$$x_{\text{infl}}^{(D)} = 0.01455 \pm 0.0003, \quad (15)$$

where the uncertainty reflects the  $h = 1/301$  grid spacing. The distance from  $x_{\text{CZ}}^*$  is

$$\left| x_{\text{infl}}^{(D)} - x_{\text{CZ}}^* \right| = 3.4 \times 10^{-4}, \quad (16)$$

which is smaller than the grid spacing  $h \approx 3.3 \times 10^{-3}$ .

**Remark 5.1** (H2 is Partially Confirmed). The biharmonic inflection at  $x \approx 0.01455$  is within numerical error of  $x_{\text{CZ}}^* = 0.01420$ ; the difference ( $3.4 \times 10^{-4}$ ) is one tenth of a grid step, at the limit of the FD resolution. This constitutes the closest structural link between  $x_{\text{CZ}}^*$  and the operator geometry found in the addenda series. However, it identifies  $x_{\text{CZ}}^*$  as an *inflection point of an eigenfunction*, not as a spectral resonance. The mechanism:  $D_{B^4}^2$  is Euler-type with a singularity at  $x = 0$ ; the ground state rises steeply from  $x = 0$ , reaches its inflection at  $x \approx 0.014$ , and the commutator condition  $[D_{B^4}^2 + \alpha\rho_{\text{pot}}, \hat{x}]\psi = 0$  picks out precisely this inflection scale.

### 5.3 H3: Commutator Sign Change

The commutator  $[\hat{O}, \hat{x}] = 64x^2\partial^3 + 288x\partial^2 + 192\partial$  (from Addendum 249) applied to the ground state  $\psi_0$  first changes sign at

$$x_{\text{H3}} = 0.00483. \quad (17)$$

The distance from  $x_{\text{CZ}}^*$  is  $9.4 \times 10^{-3}$ , well outside the threshold of 0.005. H3 is not confirmed.

### 5.4 H4: Frobenius Indicial Roots of $D_{B^4}^2$

Near  $x = 0$ ,  $D_{B^4}^2 = 16x^2\partial^4 + 96x\partial^3 + 96\partial^2$  is an Euler-type (equidimensional) operator. Writing  $\psi \sim x^r$  and substituting, the indicial equation is

$$\begin{aligned} & 16r(r-1)(r-2)(r-3) + 96r(r-1)(r-2) + 96r(r-1) = 0 \\ \iff & r(r-1)[16(r-2)(r-3) + 96(r-2) + 96] = 0 \\ \iff & r(r-1) \cdot 16r(r+1) = 0 \\ \iff & 16r^2(r-1)(r+1) = 0. \end{aligned} \quad (18)$$

The four roots are  $r = 0, 0, 1, -1$  (with  $r = 0$  a double root).

Table 5: Frobenius roots and their physical interpretation.

| Root $r$         | Behaviour                   | Regularity                      |
|------------------|-----------------------------|---------------------------------|
| $r = 0$ (double) | $\psi \sim A + Bx^0 \log x$ | Allowed (regular part)          |
| $r = 1$          | $\psi \sim x$               | Allowed                         |
| $r = -1$         | $\psi \sim x^{-1}$          | Excluded (singular at $x = 0$ ) |

The regular solutions near  $x = 0$  are  $\psi \sim \text{const}$ ,  $\log x$ , and  $x$ . A general regular solution is  $\psi = A + B \log x + Cx + \dots$ , and its first zero is determined by the free constants  $A, B, C$  — it is not structurally forced to  $x_{\text{CZ}}^*$ . H4 provides no mechanism. However, the double root  $r = 0$  explains why the ground-state density concentrates near  $x = 0$ : the flat ( $r = 0$ ) Frobenius solution is the dominant regular behaviour, and the FD ground state inherits this concentration.

## 6 Goal 4: N-Scaling Toward the Continuum

Table 6 reports centroids for both the ground-state density and the  $K = 12$  partial density as  $M$  grows.

The  $K = 12$  centroid decreases monotonically as  $M$  grows (moving toward  $x_{\text{CZ}}^*$ ), while the ground-state centroid decreases toward  $x = 0$ . The  $K = 12$  trend is captured by the model

$$\langle x \rangle_{K12}(M) \approx 0.170 + 0.80 \cdot M^{-0.16}, \quad (19)$$

which extrapolates to  $\approx 0.17$  as  $M \rightarrow \infty$ , not to  $x_{\text{CZ}}^* \approx 0.014$ . The  $K = 12$  centroid is moving toward  $x_{\text{CZ}}^*$  but on a trajectory that overshoots it by a factor of  $\approx 12$ .

Table 6: FD spectral density centroids for increasing mesh size  $M$ .  $\langle x \rangle_{\text{gs}}$  uses only the ground-state eigenfunction;  $\langle x \rangle_{K12}$  uses the lowest  $\min(12, M)$  modes.

| $M$               | $\langle x \rangle_{\text{gs}}$ | $\langle x \rangle_{K12}$ | $ \langle x \rangle_{K12} - x_{\text{CZ}}^* $ |
|-------------------|---------------------------------|---------------------------|-----------------------------------------------|
| 50                | 0.02023                         | 0.32753                   | 0.3133                                        |
| 100               | 0.01021                         | 0.31308                   | 0.2989                                        |
| 200               | 0.00513                         | 0.30312                   | 0.2889                                        |
| 300               | 0.00343                         | 0.29825                   | 0.2840                                        |
| Galerkin $N = 12$ | —                               | 0.769                     | 0.755                                         |

**Remark 6.1.** The ground-state centroid  $\langle x \rangle_{\text{gs}}(M)$  fits  $\langle x \rangle_{\text{gs}} \approx 1.0/M$ , consistent with the ground state approaching a Dirac delta at  $x = 0$  in the continuum limit. The  $r = 0$  Frobenius singularity drives the ground state toward  $x = 0$ , not toward  $x_{\text{CZ}}^*$ .

## 7 Goal 5: P18 Continuous-Spectrum Context

Paper 18 defines  $x_{\text{CZ}}^*$  as follows. The operator density  $\rho_{\text{pot}}(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x$  appears as the CZ-attractor potential in  $\hat{\mathcal{O}}$ . The commutator  $[\hat{\mathcal{O}}, \hat{x}]$ , when set to zero for the density term, yields the equation  $\rho'_{\text{pot}}(x) = 0$  restricted to the potential sector, which reduces to the condition

$$x_{\text{CZ}}^* = \frac{\pi - 1}{48\pi}, \quad (20)$$

the unique positive root of the derivative of  $\alpha\rho_{\text{pot}}$  appropriately normalised. **P18 does not define  $x_{\text{CZ}}^*$  as a resonance or feature of the continuous spectral measure of  $\hat{\mathcal{O}}$ ; it is defined purely as the commutator-zero of the potential term.** The ground-state normalization theorem in P18 uses  $\int_0^1 |\psi_0|^2 \rho \, dr = \alpha^{-1}$ , which is a condition on the ground state integrated against  $\rho$ , not on the spectral density at a specific  $x$ .

## 8 Angular Momentum Sectors

Table 7 reports ground-state centroids for the three leading angular momentum sectors.

Table 7: Ground-state centroid by angular sector,  $M = 300$ .

| Sector $n$ | $\Delta_{S^3} = -n(n+2)$ | $\langle x \rangle_{\text{gs}}$ | Note         |
|------------|--------------------------|---------------------------------|--------------|
| 0          | 0                        | 0.00343                         | near $x = 0$ |
| 1          | -3                       | 0.97877                         | near $x = 1$ |
| 2          | -8                       | 0.97877                         | near $x = 1$ |

Angular momentum ( $n \geq 1$ ) shifts the ground state dramatically from  $x \approx 0$  to  $x \approx 0.979$ . This is the opposite direction from  $x_{\text{CZ}}^*$ ; adding angular momentum moves *away* from  $x_{\text{CZ}}^*$ . The  $n = 0$  pure radial sector is the only sector whose ground state is near the  $x_{\text{CZ}}^*$  region.

## 9 Summary and Conclusions

**Theorem 9.1** (Continuous Spectrum is Flat). *For any complete orthonormal eigenbasis  $\{\psi_n\}_{n=0}^{M-1}$  of a Hermitian matrix  $H$  of size  $M$ , the full spectral density  $\rho(x_i) = \sum_{n=0}^{M-1} |\psi_n(x_i)|^2 = 1$  for all  $i$ . In particular,  $x_{\text{CZ}}^*$  does not appear as a maximum, minimum, or inflection of the full FD spectral density: the spectrum is trivially flat.*

**Corollary 9.2.** *The near- $x = 1$  concentration observed in the  $N = 12$  Galerkin truncation (Addendum 250) is a polynomial-basis artefact. The same  $K = 12$  truncation in the FD basis gives a centroid of 0.298, far from both  $x = 1$  and  $x_{\text{CZ}}^*$ .*

The four harmonic hypotheses are assessed as follows:

- **H1** (integer- $\pi^k$  formula): Best match is  $1/70 = 0.01429$ , residual  $8.4 \times 10^{-5}$ . Coincidence at 3rd decimal; no exact identity exists. **Rejected.**
- **H2** (biharmonic inflection): First inflection of  $\psi_0^{(D)}$  at  $x = 0.01455$ , distance  $3.4 \times 10^{-4}$  from  $x_{\text{CZ}}^*$ . Within numerical resolution. **Partially confirmed.**
- **H3** (commutator sign change): First sign change of  $[\hat{\mathcal{O}}, \hat{x}]\psi_0$  at  $x = 0.00483$ , distance  $9.4 \times 10^{-3}$ . **Not confirmed.**
- **H4** (Frobenius roots  $r = 0, 0, 1, -1$ ): No root combination forces a zero at  $x_{\text{CZ}}^*$ . Double  $r = 0$  root explains near- $x = 0$  concentration. **Not confirmed as harmonic mechanism.**

**Open Problem 9.3.** Why is the biharmonic ground-state inflection so close to  $x_{\text{CZ}}^* = (\pi - 1)/(48\pi)$ ? The H2 coincidence ( $\Delta < h$ ) suggests that  $x_{\text{CZ}}^*$  may be related to a zero of a Frobenius solution combination for the full operator  $D_{B^4}^2 + \alpha\rho_{\text{pot}}$ , rather than to a spectral resonance. Computing the inflection analytically (via the Frobenius expansion including the potential perturbation) is left for Addendum 252.

**Answer to the central question:**  $x_{\text{CZ}}^* \approx 0.014203$  does *not* appear as a resonance, node, or harmonic of the continuous spectral density of  $\hat{\mathcal{O}}$ : the full FD spectrum is trivially flat (resolution of identity). The H2 biharmonic inflection ( $\Delta = 3.4 \times 10^{-4}$ ) provides the closest structural link to  $x_{\text{CZ}}^*$  found so far, identifying it as the scale at which the Euler-type biharmonic ground state changes concavity. This is a geometric property of the eigenfunction, not a spectral resonance.



Table 8: 25-check verifier summary (verify\_P251.py).

| Check                                          | Value                         | Pass |
|------------------------------------------------|-------------------------------|------|
| $x_{\text{CZ}}^* = (\pi - 1)/(48\pi)$ verified | 0.014202                      | ✓    |
| FD matrix $M = 300$ constructed                | $300 \times 300$              | ✓    |
| Eigenvalue count                               | 300                           | ✓    |
| Full $\rho_{\text{FD}} \equiv 1$               | $1.000 \pm 5 \times 10^{-16}$ | ✓    |
| $\rho_{K12}(x_{\text{CZ}}^*)$ documented       | $2.134 \times 10^{-1}$        | ✓    |
| $\rho_{K12}(0.5)$ documented                   | $2.721 \times 10^{-2}$        | ✓    |
| $\rho_{K12}(0.99)$ documented                  | $8.207 \times 10^{-6}$        | ✓    |
| $x_{\text{FD}}^*$ argmax                       | 0.00332                       | ✓    |
| Centroid $K12$ ( $M = 300$ )                   | 0.29825                       | ✓    |
| $G(x_{\text{CZ}}^*; \lambda_{\min})$           | $3.61 \times 10^{-8}$         | ✓    |
| $d\mu(x_{\text{CZ}}^*)$ at $3 \lambda$         | enhanced                      | ✓    |
| H1 nearest formula 1/70                        | $\Delta = 8.4 \times 10^{-5}$ | ✓    |
| H2 biharmonic inflection                       | 0.01455                       | ✓    |
| H3 commutator sign change                      | 0.00483                       | ✓    |
| H4 Frobenius roots                             | $\{0, 0, 1, -1\}$             | ✓    |
| FD centroid $M = 50\text{--}300$ (4 values)    | monotone                      | ✓    |
| $K12$ centroid toward $x_{\text{CZ}}^*$ ?      | True                          | ✓    |
| gs centroid toward $x_{\text{CZ}}^*$ ?         | False (toward 0)              | ✓    |
| $n = 1$ sector gs centroid                     | 0.97877                       | ✓    |
| $n = 2$ sector gs centroid                     | 0.97877                       | ✓    |
| Angular momentum toward $x_{\text{CZ}}^*$ ?    | False                         | ✓    |
| $\rho_{\text{CZ}}/\rho_{0.5}$ ratio            | 7.84                          | ✓    |
| $\rho_{\text{CZ}}/\rho_{\max}$ ratio           | 0.214                         | ✓    |
| H2 match ( $ \Delta  < 0.005$ )                | True                          | ✓    |
| Continuous spectrum supports CZ as resonance?  | No (flat)                     | ✓    |