

Can x_{CZ}^* Be Identified as a Fixed Point of the Spectral Density of $\hat{\mathcal{O}}$?

Addendum 250 to the TOE Corpus

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Abstract

Addendum 249 showed that the $w = x$ self-adjointness condition for $D_{B^4}^2$ does not force $x_\infty^* = x_{\text{CZ}}^*$ — the commutator-zero subspace forces $x^* = 2/3$, not the Cauchy-zero. Here we pursue the complementary spectral approach: if $x_{\text{CZ}}^* \approx 0.01420$ is the “inward falling from every direction” attractor, the spectral density $\rho(x) = \sum_n |\psi_n(x)|^2$ should be maximised there. We compute $\rho(x)$, the spectral flow $J(x) = \sum_n |\psi_n(x)|^2 \langle \psi_n | [\hat{\mathcal{O}}, \hat{x}] | \psi_n \rangle$, the eigenvalue-weighted centroid x_λ^* , and exponentially-weighted positions $x_{\text{exp}}^*(\beta)$ in the $N = 12$ Gegenbauer-Galerkin truncation. The finding is a clean negative: every spectral statistic concentrates near $x = 1$ rather than x_{CZ}^* , and the spectral centroid $\langle x \rangle_\rho(N)$ diverges from x_{CZ}^* as N grows. The sole partial alignment is that $|J(x_{\text{CZ}}^*)| < |J(x_{\text{spec}}^*)|$, meaning the spectral flow at the CZ point is smaller than at the density maximum — but the absolute value ($\approx 2 \times 10^9$) is far from zero. We conclude that the “inward falling” interpretation, while physically motivated, does not survive contact with the Galerkin truncation of $\hat{\mathcal{O}}$ in the Gegenbauer basis. The open question of which basis or weight correctly exposes x_{CZ}^* as an attractor is sharpened.

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1 Context and Motivation

The Cauchy-zero $x_{\text{CZ}}^* = \frac{\pi-1}{48\pi} \approx 0.014202$ has appeared throughout the addenda series as the empirical ground-state position in the flat-weight Gegenbauer-Galerkin truncation of the master operator $\hat{\mathcal{O}}$ (Addenda 243–248). Addendum 249 asked whether the self-adjointness condition $w(x) = x$ forces this value; the answer was no.

The present addendum asks the complementary question from spectral theory: if x_{CZ}^* is genuinely the attractor of the system, the *spectral density* of $\hat{\mathcal{O}}$, defined as the diagonal of the spectral projector

$$\rho(x) = \sum_{n=0}^{N-1} |\psi_n(x)|^2, \quad (1)$$

should be concentrated near x_{CZ}^* . In a dissipative or convergent system, all trajectories relax toward the ground state; summing $|\psi_n|^2$ over all modes weights the ground-state region most heavily as higher modes oscillate away. This is the “inward falling from every direction” intuition.

We also examine the *spectral flow*

$$J(x) = \sum_n |\psi_n(x)|^2 \langle \psi_n | [\hat{\mathcal{O}}, \hat{x}] | \psi_n \rangle, \quad (2)$$

which measures the local drift of x under the operator flow. A fixed point of the flow satisfies $J(x^*) = 0$.

2 Setup: Operator and Galerkin Basis

2.1 The master operator

Following Papers 18 and 30 and Addenda 243–249, the truncated operator is

$$\hat{\mathcal{O}} = D_{B^4}^2 + \Delta_{S^3} + \gamma \Delta_{S^3} D_{B^4}^2 + \alpha \rho_{\text{pot}}(x), \quad (3)$$

with constants $\Omega = 4\pi^3 + \pi^2 + \pi$, $\alpha = 1/\Omega$, $\gamma = 3/4$, and CZ-attractor potential

$$\rho_{\text{pot}}(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x. \quad (4)$$

The Gegenbauer basis on $[0, 1]$ is $\{U_n\}_{n=0}^{N-1}$ with $U_0 = 1$, $U_1 = 4x - 2$, and three-term recurrence $U_n = (4x - 2)U_{n-1} - U_{n-2}$. Derivatives are computed by the differentiated recurrence (numerically stable). The Galerkin inner product uses flat weight $w(x) = 1$.

2.2 Eigenstates at $N = 12$

The $N = 12$ Galerkin matrix has 10 real eigenvalues after filtering negligible imaginary parts. The first six eigenvalues are

$$\lambda_0 \approx -5.11 \times 10^9, \quad \lambda_1 \approx -3.58 \times 10^7, \quad \lambda_2 \approx -0.509, \quad \lambda_3 \approx -0.509, \quad \lambda_4 \approx 3.27 \times 10^3, \quad \lambda_5 \approx 5.28 \times 10^4.$$

The extreme spread (nine orders of magnitude between λ_0 and λ_4) is characteristic of this operator: the $D_{B^4}^2$ component generates very large negative eigenvalues for the lowest modes, while the ρ_{pot} term contributes a large positive shift at higher basis indices.

Each eigenfunction $\psi_n(x) = \sum_k c_k^{(n)} U_k(x)$ is $L^2([0, 1])$ -normalised using Gauss–Legendre quadrature with $N_Q = 3000$ points.

3 Local Spectral Density $\rho(x)$

The local spectral density (1) is evaluated on a 500-point uniform grid over (0.001, 0.999). Key values at $N = 12$:

Quantity	Value	Formula
$\int_0^1 \rho(x) dx$	9.694	$\approx N_{\text{real}} = 10$
x_{spec}^* (argmax ρ)	0.99900	
$\langle x \rangle_\rho$ (centroid)	0.76931	$\int x \rho / \int \rho$
$x_{\text{min}}(\rho)$	0.303	argmin ρ
x_{CZ}^*	0.014202	$(\pi - 1)/(48\pi)$

The completeness check $\int \rho = 9.694 \approx 10$ confirms the normalisation is consistent with the number of real eigenstates ($< 5\%$ error). However, the maximum of $\rho(x)$ is at $x \approx 0.999$ — far from $x_{\text{CZ}}^* \approx 0.014$.

Ground-state density. The ground-state contribution $\rho_0(x) = |\psi_0(x)|^2$ has its maximum also at $x \approx 0.999$, with centroid $\langle x \rangle_{\rho_0} = 0.977$. The ground state concentrates near $x = 1$, not near x_{CZ}^* .

Eigenvalue-weighted density. The eigenvalue-weighted density $\sigma(x) = \sum_n \lambda_n |\psi_n(x)|^2$ has its $|\sigma|$ centroid at 0.975, also near $x = 1$.

Remark 3.1. The concentration of all eigenstates near $x = 1$ reflects the structure of $D_{B^4}^2 = 16x^2\partial^4 + 96x\partial^3 + 96\partial^2$ on $[0, 1]$: the fourth-order operator with boundary condition at $x = 0$ (ball boundary ∂B^4) and $x = 1$ (equator) forces large-amplitude modes near $x = 1$ in the Gegenbauer basis, which lacks an explicit zero at $x = 1$.

4 Spectral Flow $J(x)$

4.1 Commutator matrix

The commutator $[\hat{\mathcal{O}}, \hat{x}]$ is computed in the Galerkin representation as

$$[H, X]_{mn} = (HX - XH)_{mn}, \quad X_{mn} = \frac{\int_0^1 U_m(x) x U_n(x) dx}{\int_0^1 U_m^2 dx}. \quad (5)$$

The diagonal elements $\langle \psi_n | [\hat{\mathcal{O}}, \hat{x}] | \psi_n \rangle$ are

$$+4.14 \times 10^8, \quad -3.70 \times 10^8, \quad -3.32 \times 10^6, \quad -3.32 \times 10^6, \quad \dots$$

The spectral flow $J(x)$ ranges over $[-1.72 \times 10^{10}, 1.27 \times 10^{10}]$, with a single sign change (zero) at

$$x_J^* = 0.99519.$$

4.2 Comparison with x_{CZ}^*

Point	$ J(x) $	Notes
$x_{\text{CZ}}^* = 0.01420$	2.03×10^9	candidate fixed point
$x_{\text{spec}}^* = 0.99900$	1.27×10^{10}	density maximum
$x_J^* = 0.99519$	≈ 0	true J-zero

Although $|J(x_{\text{CZ}}^*)| < |J(x_{\text{spec}}^*)|$ by a factor of ≈ 6 , the absolute value at x_{CZ}^* remains large ($\approx 2 \times 10^9$). The true zero of J is the sign change at $x_J^* = 0.995$, which is near the density maximum but not near x_{CZ}^* .

Remark 4.1. The partial ordering $|J(x_{\text{CZ}}^*)| < |J(x_{\text{spec}}^*)|$ is consistent with x_{CZ}^* being in a “quieter” region of the spectral flow — the density is low there, so the flow magnitude is suppressed. This is a consequence of $\rho(x_{\text{CZ}}^*) \ll \rho(x_{\text{spec}}^*)$, not of x_{CZ}^* being a genuine fixed point.

5 N -Scaling of the Spectral Centroid

N	$\langle x \rangle_\rho(N)$	$ \langle x \rangle_\rho - x_{\text{CZ}}^* $
6	0.60907	0.59487
8	0.67139	0.65719
10	0.72424	0.71004
12	0.76931	0.75511
14	0.79075	0.77655
16	0.80324	0.78904

The spectral centroid *diverges* from x_{CZ}^* as N grows, increasing monotonically from 0.609 at $N = 6$ to 0.803 at $N = 16$. The centroid is converging toward $x \approx 1$, not toward $x_{\text{CZ}}^* \approx 0.014$. The ground-state N -scaling series (Addendum 248) showed slow drift toward x_{CZ}^* in the ground-state position; the spectral centroid moves in the opposite direction.

Remark 5.1. The divergence of the spectral centroid from x_{CZ}^* does not contradict Addendum 248. The spectral centroid $\langle x \rangle_\rho$ averages over *all* eigenstates with equal weight. As N grows, more high-energy modes (concentrated near $x = 1$) enter the sum and drag the centroid rightward. The ground-state centroid $\langle x \rangle_{\rho_0}$ is the relevant quantity for the x_{CZ}^* hypothesis, and it also concentrates near $x = 1$ in this basis at $N = 12$.

6 Eigenvalue-Weighted and Exponentially-Weighted Positions

6.1 Eigenvalue-weighted centroid

The eigenvalue-weighted centroid

$$x_\lambda^* = \frac{\sum_n \lambda_n \langle \psi_n | x | \psi_n \rangle}{\sum_n \lambda_n} \quad (6)$$

evaluates to $x_\lambda^* = 0.97630$ at $N = 12$. The large negative ground-state eigenvalue ($\lambda_0 \approx -5.1 \times 10^9$) dominates, and since the ground state concentrates near $x = 1$, the weighted centroid is also near $x = 1$.

6.2 Exponential weighting

The exponentially-weighted centroid $x_{\text{exp}}^*(\beta)$ is defined by

$$x_{\text{exp}}^*(\beta) = \frac{\sum_n e^{-\beta \lambda_n} \langle \psi_n | x | \psi_n \rangle}{\sum_n e^{-\beta \lambda_n}}. \quad (7)$$

β	$x_{\text{exp}}^*(\beta)$	$ x_{\text{exp}}^* - x_{\text{CZ}}^* $
0.001	0.97856	0.96436
0.01	0.97856	0.96436
0.1	0.97856	0.96436
1.0	0.97856	0.96436

The $x_{\text{exp}}^*(\beta)$ is constant across all tested β values because the dominant exponential weight always belongs to $\lambda_0 \approx -5.1 \times 10^9$ (shifting eigenvalues by $-\lambda_{\min}$ before exponentiation, the ground state weight $e^0 = 1$ while all others are $e^{-\beta(\lambda_n - \lambda_0)} \approx 0$ for any $\beta > 0$). The function $x_{\text{exp}}^*(\beta)$ does *not* cross x_{CZ}^* for any positive β .

7 Relation to Paper 18 and the Operator Foundation

Paper 18 defines $\hat{\mathcal{O}}$ and states that the eigenvalue equation $\hat{\mathcal{O}}\psi_n = E_n\psi_n$ encodes fundamental physics through the ground-state normalisation (α^{-1}), the spectral gap (M_{Pl}), and the discrete spectrum (fermion masses). The paper defines $\rho(x)$ as the geometric density $16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x$ and establishes the potential coupling. The value $x_{\text{CZ}}^* = (\pi - 1)/(48\pi)$ is not defined in Paper 18 as a fixed point; it emerged from the Galerkin series (Addendum 243 onward) as the empirical ground-state position in the flat-weight truncation.

The flow-zero interpretation — that x_{CZ}^* is the point where “inward falling terminates” — is consistent with Paper 18’s framing of $\rho_{\text{pot}}(x)$ as a self-lensing attractor, but the spectral density analysis shows this interpretation is not borne out quantitatively in the $N = 12$ Gegenbauer truncation.

8 Conclusions and Open Questions

Theorem 8.1 (Spectral density does not concentrate at x_{CZ}^*). *In the $N = 12$ Gegenbauer-Galerkin truncation of $\hat{\mathcal{O}}$, with flat weight $w(x) = 1$:*

- (i) *The local spectral density $\rho(x) = \sum_n |\psi_n(x)|^2$ is maximised at $x_{\text{spec}}^* \approx 0.999$, not at x_{CZ}^* .*
- (ii) *The spectral centroid $\langle x \rangle_\rho \approx 0.769$ and diverges from x_{CZ}^* as N grows.*
- (iii) *The spectral flow $J(x)$ has its zero at $x^* \approx 0.995$, not at x_{CZ}^* .*
- (iv) *The eigenvalue-weighted position $x_\lambda^* \approx 0.976$ and $x_{\text{exp}}^*(\beta) \approx 0.979$ for all tested β .*
- (v) *None of the above quantities crosses or converges to $x_{\text{CZ}}^* \approx 0.014$.*

Partial alignment. The spectral flow at the CZ point satisfies $|J(x_{\text{CZ}}^*)| \approx 2.0 \times 10^9 < |J(x_{\text{spec}}^*)| \approx 1.3 \times 10^{10}$, a factor-6 suppression. This is consistent with x_{CZ}^* being in a low-density region of the flow, but does not constitute proximity to a fixed point.

Open Problem 8.2. Identify the inner product (weight $w(x)$), basis, or regularisation under which the spectral density of $\hat{\mathcal{O}}$ is maximised at x_{CZ}^* , or prove that no such choice exists within the family of polynomial Galerkin truncations.

Open Problem 8.3. The ground-state position $\langle x \rangle_{\rho_0}(N)$ drifts toward x_{CZ}^* under the Jacobi weight (Addendum 248) but the spectral centroid $\langle x \rangle_\rho(N)$ diverges from it under flat weight. Determine whether any convex combination of eigenstate contributions $\sum_n \alpha_n \langle \psi_n | x | \psi_n \rangle$ with $\alpha_n \geq 0$ and $\sum_n \alpha_n = 1$ converges to x_{CZ}^* as $N \rightarrow \infty$.

Summary. The series A246–A250 constitutes a systematic elimination: flat-weight ground state (A246–A248) converges slowly to x_{CZ}^* under Jacobi weight; $w = x$ self-adjointness (A249) forces a different fixed point ($x = 2/3$); spectral density (A250) concentrates near $x = 1$. The status of x_{CZ}^* as a spectral attractor of $\hat{\mathcal{O}}$ remains an open problem requiring either a different inner product or a different definition of “attractor.”

Numerical Setup

All computations use `numpy` for matrix algebra and `mpmath` (dps=60) for high-precision constants. Gauss–Legendre quadrature with $N_Q = 3000$ points on $[0, 1]$. The full verifier with 30 checks is at `Lumen/corpus/addenda/verify/verify_P250.py`.

Checks passed: 30/30.

Constants: $x_{\text{CZ}}^* = 0.01420188\dots$; $\Omega = 4\pi^3 + \pi^2 + \pi \approx 137.036$; $\alpha = 1/\Omega$; $\gamma = 3/4$.