

Does $w = x$ Self-Adjointness of $D_{B^4}^2$
Force $x_\infty^* = x_{CZ}^*$?
Commutator Analysis and $w = x$ Galerkin
Addendum 249 to the TOE Corpus

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Abstract

Addendum 248 (A248) established that $D_{B^4}^2 = 16x^2\partial^4 + 96x\partial^3 + 96\partial^2$ is formally self-adjoint under the unique weight $w(x) = x$ (Sturm-Liouville eigenvalue $\lambda = 0$, Euler ODE $x^2w'' + xw' - w = 0$). This addendum answers the question: does that self-adjointness condition structurally force $x_\infty^* = x_{CZ}^* = (\pi - 1)/(48\pi)$?

Answer: No. The commutator $[\hat{O}, \hat{x}]$ is computed explicitly. For the ground mode $n = 0$, $[\hat{O}, \hat{x}] = 64x^2\partial^3 + 288x\partial^2 + 192\partial$, a third-order operator. The commutator-zero equation $[\hat{O}, \hat{x}]\psi = 0$ reduces to the Euler ODE $x^2u'' + \frac{9}{2}xu' + 3u = 0$ (with $u = \psi'$), whose indicial roots are $r_{1,2} = -\frac{3}{2}, -2$ (both negative). The only regular solution on $[0, 1]$ is $\psi = \text{const}$, which gives $x^*(w=x) = \frac{2}{3}$ — far from $x_{CZ}^* \approx 0.01420$. The $w = x$ Galerkin computation (Section 5) confirms this structurally: $x^*(N)$ under $w = x$ is monotone *increasing* from 0.9627 at $N = 6$ to 0.9956 at $N = 20$, extrapolating to $x_\infty^* \approx 0.9995$, which is $69 \times x_{CZ}^*$. The $w = x$ inner product emphasises large x , pulling the ground state toward the far end of $[0, 1]$; it is the *wrong* weight for locating x_{CZ}^* . All 30 checks in `verify_P249.py` pass at `mpmath dps = 60`.

1 Setup and Context

The master operator is

$$\hat{O} = D_{B^4}^2 + \Delta_{S^3} + \gamma T + \alpha \rho(x), \quad T = \Delta_{S^3} \circ D_{B^4}^2, \quad (1)$$

where $D_{B^4}^2 = 16x^2\partial^4 + 96x\partial^3 + 96\partial^2$ acts on $[0, 1]$; Δ_{S^3} is diagonal with mode- n eigenvalue $-n(n+2)$; $\gamma = 3/4$; $\alpha = 1/\Omega$ with $\Omega = 4\pi^3 + \pi^2 + \pi$; and $\rho(x) = 16\pi^3x^3 + 3\pi^2x^2 + 2\pi x$. The commutator-zero fixed point is

$$x_{CZ}^* = \frac{\pi - 1}{48\pi} \approx 0.014202. \quad (2)$$

A248 proved that the unique weight making $D_{B^4}^2$ formally self-adjoint is $w(x) = x$ (satisfying the Euler ODE $x^2w'' + xw' - w = 0$ with Sturm-Liouville eigenvalue $\lambda = 0$). The question is whether this geometric fact forces the ground-state expectation $x^* \rightarrow x_{CZ}^*$.

2 Commutator $[\hat{O}, \hat{x}]$ Analysis

2.1 Commutator of $D_{B^4}^2$ with $\hat{x}$

For a differential operator L and the multiplication operator $\hat{x}$, the commutator $[L, \hat{x}]g = L(xg) - x \cdot Lg$ is computed by expanding $(xg)^{(k)}$ via the Leibniz rule: $(xg)' = xg' + g$, $(xg)'' = xg'' + 2g'$, $(xg)''' = xg''' + 3g''$, $(xg)'''' = xg'''' + 4g'''$.

Proposition 2.1 (Commutator of $D_{B^4}^2$ with $\hat{x}$).

$$[D_{B^4}^2, \hat{x}] = 64x^2\partial^3 + 288x\partial^2 + 192\partial. \quad (3)$$

On monomials x^n , this yields

$$[D_{B^4}^2, \hat{x}]x^n = 32n(n+1)(2n+1)x^{n-1}. \quad (4)$$

Proof. Expanding $D_{B^4}^2(xg)$ and subtracting $x \cdot D_{B^4}^2g$:

$$\begin{aligned} D_{B^4}^2(xg) &= 16x^2(xg'''' + 4g''') + 96x(xg''' + 3g'') + 96(xg'' + 2g') \\ &= x \cdot D_{B^4}^2g + 64x^2g''' + 288xg'' + 192g'. \end{aligned}$$

For $g = x^n$: inserting $g''' = n(n-1)(n-2)x^{n-3}$, $g'' = n(n-1)x^{n-2}$, $g' = nx^{n-1}$ gives $64n(n-1)(n-2) + 288n(n-1) + 192n = n[64n^2 + 96n + 32] = 32n(2n+1)(n+1)$. $\square$

Numerically verified: $[D_{B^4}^2, \hat{x}]x^n$ for $n = 1, 2, 3, 4$ gives 192, 960, 2688, 5760, matching $32n(n+1)(2n+1)$ exactly (checks C01–C04).

2.2 Full commutator $[\hat{\mathcal{O}}, \hat{x}]$

Proposition 2.2 (Full commutator of $\hat{\mathcal{O}}$ with $\hat{x}$). *For a fixed mode n (where Δ_{S^3} acts as the scalar $\lambda_n = -n(n+2)$):*

$$[\hat{\mathcal{O}}, \hat{x}] = (1 + \gamma\lambda_n) [D_{B^4}^2, \hat{x}]. \quad (5)$$

In particular, for the ground mode $n = 0$ ($\lambda_0 = 0$):

$$[\hat{\mathcal{O}}, \hat{x}]|_{n=0} = 64x^2\partial^3 + 288x\partial^2 + 192\partial. \quad (6)$$

Proof. Three terms contribute to $[\hat{\mathcal{O}}, \hat{x}]$:

- (i) $[\rho(x)\cdot, \hat{x}] = 0$: multiplication operators commute.
- (ii) $[\Delta_{S^3}, \hat{x}] = [\lambda_n \cdot \text{Id}, \hat{x}] = 0$: in mode n , Δ_{S^3} is the scalar λ_n , which commutes with everything.
- (iii) $[T, \hat{x}] = [\Delta_{S^3} \circ D_{B^4}^2, \hat{x}] = \Delta_{S^3} \circ [D_{B^4}^2, \hat{x}]$ (since $\Delta_{S^3} = \lambda_n \cdot \text{Id}$ in mode n) $= \lambda_n [D_{B^4}^2, \hat{x}]$.

Summing: $[\hat{\mathcal{O}}, \hat{x}] = [D_{B^4}^2, \hat{x}] + \gamma\lambda_n [D_{B^4}^2, \hat{x}] = (1 + \gamma\lambda_n) [D_{B^4}^2, \hat{x}]$. $\square$

Remark 2.3. For mode $n = 1$: $\lambda_1 = -3$, factor $= 1 - 9/4 = -5/4$ (check C08). For mode $n = 2$: $\lambda_2 = -8$, factor $= 1 - 6 = -5$ (check C26). The commutator grows more negative for higher modes; the ground mode $n = 0$ has the simplest form.

3 Commutator-Zero Subspace

In mode $n = 0$, the commutator-zero condition $[\hat{\mathcal{O}}, \hat{x}]\psi = 0$ is

$$64x^2\psi''' + 288x\psi'' + 192\psi' = 0. \quad (7)$$

Setting $u = \psi'$ yields the Euler ordinary differential equation

$$x^2u'' + \frac{9}{2}xu' + 3u = 0. \quad (8)$$

Theorem 3.1 (Commutator-zero subspace is trivial). *The only function $\psi \in C^3([0, 1])$ that is regular at $x = 0$ and satisfies (7) is $\psi = \text{const}$.*

Proof. The Euler ODE (8) has the ansatz $u = x^r$, giving the indicial equation

$$r(r-1) + \frac{9}{2}r + 3 = r^2 + \frac{7}{2}r + 3 = 0. \quad (9)$$

The discriminant is $49/4 - 12 = 1/4$, yielding

$$r_{1,2} = \frac{-7/2 \pm 1/2}{2} = -\frac{3}{2}, -2. \quad (10)$$

Both roots are negative; the general solution is $u(x) = Ax^{-3/2} + Bx^{-2}$, which is singular at $x = 0$ unless $A = B = 0$. Hence $u = \psi' = 0$, i.e. $\psi = \text{const}$. $\square$

Corollary 3.2 (Commutator-zero x^* under $w = x$). *The commutator-zero regular ground state $\psi = 1$ has*

$$x_{w=x}^*(\text{const}) = \frac{\int_0^1 x \cdot 1^2 \cdot x \, dx}{\int_0^1 1^2 \cdot x \, dx} = \frac{1/3}{1/2} = \frac{2}{3} \approx 0.667. \quad (11)$$

This is 47 times larger than $x_{CZ}^ \approx 0.01420$. The $w = x$ self-adjointness condition does not force $x_\infty^* = x_{CZ}^*$.*

Remark 3.3 (Why x_{CZ}^* and commutator-zero are different conditions). x_{CZ}^* is the Cauchy-zero of the $\hat{\mathcal{O}}$ operator density (A243): the x -value where a certain spectral density function passes through zero. Theorem 3.1 concerns a completely different condition: the subspace of functions whose position expectation is *sharp* (i.e. $\hat{x}$ commutes with $\hat{\mathcal{O}}$ on that function). The two conditions coincide only if the ground state is simultaneously an eigenstate of $\hat{x}$, which requires $\hat{x}$ to commute with $\hat{\mathcal{O}}$ — and the only regular such state is a constant, giving $x^* = 2/3$, not x_{CZ}^* .

4 $w = x$ Inner Product Matrix

4.1 Monomial matrix elements

Under the $w = x$ inner product $\langle f, g \rangle_x = \int_0^1 f(x)g(x)x \, dx$, the matrix elements of $D_{B^4}^2$ in the monomial basis $\{x^n\}$ are

$$M[m, n] = \int_0^1 x^m [D_{B^4}^2(x^n)] x \, dx = \frac{16n^2(n^2 - 1)}{m + n}, \quad n \geq 2; \quad M[m, n] = 0 \text{ for } n < 2. \quad (12)$$

The Gram matrix is

$$G[m, n] = \int_0^1 x^{m+n+1} \, dx = \frac{1}{m + n + 2}. \quad (13)$$

Both formulas are verified for $m, n = 0, 1, 2, 3$ at 10^{-10} precision (checks C12–C13).

4.2 Jacobi orthonormal basis under $w = x$

The Jacobi polynomials $P_n^{(0,1)}$ on $[-1, 1]$ (weight $(1+t)$) shifted to $[0, 1]$ via $t = 2x - 1$ are orthogonal under $x \, dx$. Their norms satisfy

$$\int_0^1 [P_n^{(0,1)}(2x - 1)]^2 x \, dx = \frac{1}{2(n+1)}, \quad (14)$$

so the orthonormal basis is $\varphi_n(x) = \sqrt{2(n+1)} P_n^{(0,1)}(2x - 1)$. Orthonormality $\langle \varphi_m, \varphi_n \rangle_x = \delta_{mn}$ is verified for $m, n = 0, \dots, 4$ (25 pairs, check C14).

5 $w = x$ Galerkin Computation

5.1 Setup

The Galerkin matrix is built using the same Gegenbauer basis as A246/A248 but with the $w = x$ inner product for all matrix entries. For a basis function U_m :

$$h_m = \int_0^1 U_m^2 x \, dx, \quad (15)$$

$$[D_2^{(wx)}]_{mn} = \frac{1}{h_m} \int_0^1 U_m(x) [D_{B^4}^2 U_n](x) x \, dx, \quad (16)$$

$$[D_\Delta]_{mn} = -n(n+2) \delta_{mn} \quad (\text{mode eigenvalue}), \quad (17)$$

$$H^{(wx)} = D_2^{(wx)} + D_\Delta + \gamma D_\Delta D_2^{(wx)} + \alpha \rho^{(wx)}. \quad (18)$$

The ground-state expectation value is

$$x^*(N; w=x) = \frac{\int_0^1 x^2 |\psi_0|^2 \, dx}{\int_0^1 x |\psi_0|^2 \, dx} = \frac{\langle x \rangle_{w=x}^{\text{num}}}{\langle 1 \rangle_{w=x}^{\text{den}}}. \quad (19)$$

This is the $w = x$ expectation of $\hat{x}$, equal to $\langle x^2 \rangle_{\text{flat}} / \langle x \rangle_{\text{flat}}$. All integrals use $N_q = 4000$ -point Gauss-Legendre quadrature.

5.2 Results

Table 1: $x^*(N; w=x)$ computed by $w = x$ Galerkin diagonalisation.

N	$x^*(N; w=x)$	$ x^* - x_{\text{CZ}}^* $	drift
6	0.96273116	0.94852929	—
8	0.97801486	0.96381298	+0.01528
10	0.98518057	0.97097869	+0.00717
12	0.98923229	0.97503041	+0.00405
14	0.99177729	0.97757542	+0.00255
16	0.99349177	0.97928989	+0.00171
18	0.99470685	0.98050497	+0.00122
20	0.99560202	0.98140014	+0.00090

All seven consecutive steps are *monotone increasing*. The sequence moves away from x_{CZ}^* as N grows. A power-law fit $x^*(N) = x_\infty^* + c/N^\alpha$ gives $x_\infty^* \approx 0.9995$ (RMSE 5.6×10^{-5} , $\alpha \approx 1.85$), which is $70 \times x_{\text{CZ}}^*$.

Table 2: $x^*(N = 12)$ under three inner product weights.

Weight	$x^*(N = 12)$	Note
$w = x$	0.9892	This addendum; converges to ≈ 1
$w = (1 - x)^{3/2}$ (Haar exp.)	0.0955	A248, monotone decreasing
x_{CZ}^*	0.0142	Target fixed point

5.3 Interpretation

Under $w = x$ the inner product assigns weight proportional to x , so the operator matrix heavily favours the region $x \approx 1$. The ground state of $H^{(wx)}$ concentrates near $x = 1$, giving $x^* \rightarrow 1$ as $N \rightarrow \infty$. This is geometrically opposite to $x_{\text{CZ}}^* \approx 0.014$, which sits near $x = 0$.

By contrast, the $w = (1 - x)^{3/2}$ weight (A248) assigns most weight near $x = 0$, and the corresponding $x^*(N)$ decreases monotonically toward x_{CZ}^* — though still far from it at $N = 20$.

The Galerkin result thus confirms the analytic conclusion of Section 3: $w = x$ is the *wrong* weight for locating x_{CZ}^* . Its role in A248 was purely structural (SL self-adjointness of $D_{B^4}^2$), not physical (locating the fixed point).

5.4 $D_{B^4}^2$ alone under $w = x$

For completeness, $D_{B^4}^2$ alone (no Δ_{S^3} , T , or ρ terms) under $w = x$ at $N = 20$ gives $x^*(D^2; w=x; N=20) = 0.98885$, consistent with the full- $\hat{\mathcal{O}}$ result. The $\rho(x)$ potential term contributes only a small correction ($\Delta x^* \approx 0.007$ at $N = 20$), confirming that the dominant effect is the $w = x$ weighting geometry, not the potential.

6 Structural Explanation: Why x_{CZ}^* Is Not Forced

The following diagram summarises the logical chain:

1. **A248 result:** $w = x$ is the unique weight making $D_{B^4}^2$ self-adjoint (SL eigenvalue $\lambda = 0$, Euler ODE).
2. **Commutator-zero condition** ($[\hat{\mathcal{O}}, \hat{x}]\psi = 0$): forces $\psi = \text{const}$ by Theorem 3.1.
3. x^* **for constant ψ under $w = x$:** equals $2/3$ (Corollary 3.2).
4. $x_{\text{CZ}}^* = 2/3$? No; $x_{\text{CZ}}^* \approx 0.014 \neq 2/3$.
5. $w = x$ **Galerkin:** $x^*(N) \rightarrow 1$, moving away from x_{CZ}^* .

The chain breaks at step 4. x_{CZ}^* is not the expectation of $\hat{x}$ for any commutator-zero state; it enters from a different structural source (the Cauchy-zero of the spectral density). The $w = x$ self-adjointness of $D_{B^4}^2$ is a real fact, but it locates neither x_{CZ}^* nor the physical ground state.

7 Summary

Table 3: A249 key results.

Quantity	Value
$[D_{B^4}^2, \hat{x}](x^n)$	$32n(n+1)(2n+1)x^{n-1}$ (exact)
$[D_{B^4}^2, \hat{x}]$ as operator	$64x^2\partial^3 + 288x\partial^2 + 192\partial$
$[\hat{\mathcal{O}}, \hat{x}]$ in mode n	$(1 + \frac{3}{4}\lambda_n)[D_{B^4}^2, \hat{x}]$
Comm.-zero ODE indicial roots	$r_1 = -3/2, r_2 = -2$
Regular null space dimension	1 (only $\psi = \text{const}$)
$x^*(\psi=\text{const}; w=x)$	$2/3 \approx 0.667$
x_{CZ}^*	$(\pi - 1)/(48\pi) \approx 0.01420$
$w = x$ forces $x_\infty^* = x_{\text{CZ}}^*$?	No
$x^*(D_{B^4}^2; w=x; N=20)$	0.98885
$x^*(\hat{\mathcal{O}}; w=x; N=12)$	0.98923
$x^*(\hat{\mathcal{O}}; w=x; N=20)$	0.99560
$x_\infty^*(w=x)$ extrapolated	≈ 0.9995
Convergence direction under $w = x$	Increasing (away from x_{CZ}^*)
Comparison weight at $N = 12$	$x^*(w=(1-x)^{3/2}) = 0.0955$ (A248)
Verifier checks	30/30 PASS

Conclusions. (1) $[\hat{\mathcal{O}}, \hat{x}]$ in the ground mode is $64x^2\partial^3 + 288x\partial^2 + 192\partial$. (2) Its null space on regular functions is one-dimensional, spanned by $\psi = \text{const}$. (3) The constant function gives $x^*(w=x) = 2/3$, not x_{CZ}^* . (4) The $w = x$ Galerkin confirms this: $x^*(N)$ increases monotonically to ≈ 1 under $w = x$, the geometric opposite of $x_{\text{CZ}}^* \approx 0.014$. (5) The $w = x$ self-adjointness of $D_{B^4}^2$ is a valid SL fact but does not link to x_{CZ}^* , which arises from the Cauchy-zero structure of the spectral density (A243), not from commutator algebra.

8 Verifier

All claims are certified by `Lumen/corpus/addenda/verify/verify_P249.py` (30 independent checks at `mpmath dps=60`): commutator algebra C01–C08, commutator-zero subspace C09–C11, inner product matrices C12–C14, $D_{B^4}^2$ -alone ground state C15–C16, full Galerkin C17–C19, structural conclusion C20–C22, and sanity checks C23–C30. All 30 pass.

References

- [1] L. F. Vlegels, *Addendum 242: $\hat{\mathcal{O}}$ Eigentrajectory and x_{CZ}^* Fixed Point*, TOE Corpus, 2026.
- [2] L. F. Vlegels, *Addendum 243: Gegenbauer Basis Galerkin Approach to x^** , TOE Corpus, 2026.
- [3] L. F. Vlegels, *Addendum 246: Unbiased Stable Basis for $\hat{\mathcal{O}}$ Eigentrajectory*, TOE Corpus, 2026.
- [4] L. F. Vlegels, *Addendum 248: Sturm-Liouville Weight Derivation and $x^*(N) \rightarrow x_{\text{CZ}}^*$ Convergence*, TOE Corpus, 2026.