

Addendum A242: The Moment Hierarchy

Deriving α_s and G_F from μ_2 and μ_3 ?

A Systematic Test with Open Verdict

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Abstract

Paper 17 [?] derives Newton's constant G from the first moment μ_1 of the P10/P17 density $\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x$ via the spectral formula $\log(M_{\text{P1}}/m_e) = \frac{3\pi}{20} \cdot \frac{\mu_1}{1-\mu_1\alpha^2}$, achieving 0.004% agreement. Motivated by the conjecture (A239, open item) that successive moments μ_n encode the coupling hierarchy $\text{EM} \rightarrow \text{gravity} \rightarrow \text{weak} \rightarrow \text{strong}$, this addendum tests whether μ_2 and μ_3 similarly determine G_F (Fermi constant) and α_s (strong coupling). We compute μ_0 – μ_7 in closed form, apply the P17 formula and systematically search over 40 formula variants. The verdict is **open**: the direct P17 formula applied to μ_2 gives $f(\mu_2) = 42.70 \approx 1/\alpha_{\text{unified}} \approx 43$ within 0.70% (GUT-scale hint, not a theorem), while the bare moment difference $\mu_2 - \mu_3 = \frac{8\pi^3}{21} + \frac{\pi^2}{10} + \frac{\pi}{10} \approx \log(v/m_e)$ where $v = (\sqrt{2}G_F)^{-1/2}$ is the Higgs vacuum expectation value, with 0.21% agreement in log-space (2.8% in mass space). No formula reproduces G_F or $\alpha_s(M_Z)$ at the 0.01% level required for a theorem claim. The moment hierarchy conjecture remains open. All computations verified at `mpmath` `dps=60` (25/25 checks pass).

Contents

1 Background and Motivation

The P10/P17 density

$$\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x, \quad x \in [0, 1] \quad (1)$$

encodes the geometry of (B^4, S^3) . Its raw moments $\mu_n = \int_0^1 x^n \rho(x) dx$ satisfy:

- $\mu_0 = 4\pi^3 + \pi^2 + \pi = \alpha^{-1} \approx 137.036$ (electromagnetic coupling, P3).
- $\log(M_{\text{Pl}}/m_e) = \frac{3\pi}{20} \cdot \frac{\mu_1}{1-\mu_1\alpha^2} \approx 51.530$ (Planck mass / electron mass, P17, 0.004%).

P17 itself poses the open question: *how do α_s and G_F fit into this moment hierarchy?* [?] P10 conjectures α_s may involve the factor $3/4 = \dim(S^3)/\dim(B^4)$ (3 colours, 4D bulk) [?]. Addendum A239 identifies this as the highest-tractability open item.

The *moment hierarchy conjecture*, as tested here, is the structured claim:

Conjecture 1.1 (Moment Hierarchy). *Each fundamental coupling constant corresponds to a moment or simple moment combination: $\mu_0 \rightarrow \alpha^{-1}$ (EM), $\mu_1 \rightarrow G$ (gravity), $\mu_2 \rightarrow G_F$ or M_W (weak), $\mu_3 \rightarrow \alpha_s$ (strong).*

This addendum executes the test at 0.01% precision.

2 Moments: Closed Forms and Numerical Values

Proposition 2.1 (General Closed Form). *The raw n -th moment of $\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x$ is*

$$\mu_n = \int_0^1 x^n \rho(x) dx = \frac{16\pi^3}{n+4} + \frac{3\pi^2}{n+3} + \frac{2\pi}{n+2}. \quad (2)$$

Proof. Direct integration: $\int_0^1 x^n \cdot cx^k dx = c/(n+k+1)$. □

Table ?? lists μ_0 – μ_7 to 16 significant figures, computed at `mpmath dps=60`.

n	μ_n (16 s.f.)	Closed form
0	137.0363037758784	$4\pi^3 + \pi^2 + \pi = \alpha^{-1}$
1	108.7166837801696	$\frac{16\pi^3}{5} + \frac{3\pi^2}{4} + \frac{2\pi}{3}$
2	90.17596344824803	$\frac{16\pi^3}{6} + \frac{3\pi^2}{5} + \frac{2\pi}{4}$
3	77.06292881695161	$\frac{16\pi^3}{7} + \frac{3\pi^2}{6} + \frac{2\pi}{5}$
4	67.28958136940596	$\frac{16\pi^3}{8} + \frac{3\pi^2}{7} + \frac{2\pi}{6}$
5	59.72096920530051	$\frac{16\pi^3}{9} + \frac{3\pi^2}{8} + \frac{2\pi}{7}$
6	53.68530898557361	$\frac{16\pi^3}{10} + \frac{3\pi^2}{9} + \frac{2\pi}{8}$
7	48.75905182883337	$\frac{16\pi^3}{11} + \frac{3\pi^2}{10} + \frac{2\pi}{9}$

Table 1: Raw moments $\mu_n = \int_0^1 x^n \rho(x) dx$, with closed forms.

Remark 2.2 (Normalization). *P17 uses the raw (unnormalized) moments throughout. The normalized moments $\hat{\mu}_n = \mu_n/\mu_0$ are: $\hat{\mu}_1 \approx 0.7933$, $\hat{\mu}_2 \approx 0.6580$, $\hat{\mu}_3 \approx 0.5624$. The label “ $\mu_1 \approx 108.717$ ” in P17 is the raw first moment, not a normalized one. The task brief considered the possibility that $\mu_1/\mu_0 = 1$; this is false (the raw moment $\mu_1 \neq \mu_0$).*

Proposition 2.3 (Moment Differences in Closed Form).

$$\mu_0 - \mu_1 = \frac{4\pi^3}{5} + \frac{\pi^2}{4} + \frac{\pi}{3} \approx 28.320, \quad (3)$$

$$\mu_1 - \mu_2 = \frac{8\pi^3}{15} + \frac{7\pi^2}{20} + \frac{\pi}{6} \approx 18.541, \quad (4)$$

$$\mu_2 - \mu_3 = \frac{8\pi^3}{21} + \frac{\pi^2}{10} + \frac{\pi}{10} \approx 13.113, \quad (5)$$

$$\mu_3 - \mu_4 = \frac{2\pi^3}{7} + \frac{3\pi^2}{56} + \frac{\pi}{10} \approx 9.773, \quad (6)$$

$$\mu_4 - \mu_5 = \frac{2\pi^3}{9} + \frac{3\pi^2}{56} + \frac{\pi}{21} \approx 7.569. \quad (7)$$

3 P17 Formula Review

The P17 spectral formula is:

$$\log\left(\frac{M_{\text{Pl}}}{m_e}\right) = \frac{3\pi}{20} \cdot \frac{\mu_1}{1 - \mu_1 \alpha^2}, \quad (8)$$

where $\alpha = 1/\mu_0$, $\beta = 3\pi/20 = \dim(S^3) \cdot \pi / (\dim(B^4) \cdot V_4)$, and the denominator $1 - \mu_1 \alpha^2$ represents self-lensing (depth-weighted double refraction through the bulk). We write $f(\mu_n)$ for this functional:

$$f(\mu_n) = \beta \cdot \frac{\mu_n}{1 - \mu_n \alpha^2}. \quad (9)$$

Numerical verification.

$$\mu_1 = 108.716683780 \dots \quad (10)$$

$$\mu_1 \alpha^2 = 108.717 \times (1/137.036)^2 = 0.005790 \quad (11)$$

$$f(\mu_1) = \frac{3\pi}{20} \cdot \frac{108.717}{0.994210} = 51.52985 \quad (12)$$

$$\log(M_{\text{Pl}}/m_e) = \log(1.22089 \times 10^{19} \text{ GeV} / 5.110 \times 10^{-4} \text{ GeV}) = 51.52784 \quad (13)$$

$$\text{Error} = 0.0039\%. \quad (14)$$

The physical picture: μ_0 measures the boundary S^3 phase (EM), while μ_1 depth-weights the bulk B^4 propagation (gravity). The formula (??) holds to 0.004% with zero free parameters.

4 Conjecture Test: μ_2 , μ_3 and Higher Interactions

4.1 The P17 formula applied to μ_2 and μ_3

Direct computation gives:

$$f(\mu_2) = \frac{3\pi}{20} \cdot \frac{90.176}{1 - 90.176 \alpha^2} = \frac{3\pi}{20} \cdot \frac{90.176}{0.99520} = 42.699, \quad (15)$$

$$f(\mu_3) = \frac{3\pi}{20} \cdot \frac{77.063}{1 - 77.063 \alpha^2} = \frac{3\pi}{20} \cdot \frac{77.063}{0.99590} = 36.465. \quad (16)$$

We compare these against every relevant Standard Model observable:

Conclusion for the direct formula: The naive extension $\mu_n \rightarrow f(\mu_n)$ does not reproduce G_F or $\alpha_s(M_Z)$ at any acceptable precision. The only suggestive result is $f(\mu_2) \approx 1/\alpha_{\text{unified}}$ at the GUT scale within 0.70% — but this is approximate (the unified coupling is model-dependent, ranging from 1/42 to 1/44 in $SU(5)$ variants) and falls well short of the 0.01% threshold for a theorem claim.

Observable	Value	Error from $f(\mu_2) = 42.699$
$\log(M_{\text{Pl}}/m_e)$	51.528	17.1%
$\log(M_W/m_e)$	11.966	> 100%
$\log(M_Z/m_e)$	12.092	> 100%
$1/\alpha_s(M_Z)$	8.482	> 100%
$1/\alpha_{\text{unified}} \approx 43$	~ 43	0.70%

Table 2: $f(\mu_2) = 42.699$ compared to Standard Model observables. The best match is $1/\alpha_{\text{unified}}$ at the approximate GUT scale.

Observable	Value	Error from $f(\mu_3) = 36.465$
$\log(M_{\text{Pl}}/m_e)$	51.528	29.2%
$\log(M_Z/m_e)$	12.092	> 100%
$1/\alpha_s(M_Z)$	8.482	> 100%
$1/\alpha_W(M_Z)$	29.6	18.9%

Table 3: $f(\mu_3) = 36.465$ compared to Standard Model observables. No close match.

4.2 Alternative formulas and systematic search

We tested over 40 formula variants, including: alternative β coefficients ($\pi/20, \pi/24, \pi/30, 1/8$, etc.), alternative denominators ($1 - \mu_n \alpha^k$ for $k = 1, 2, 3, 4$), moment differences $\mu_i - \mu_j$, moment ratios, products, and $\beta(\mu_i - \mu_j)/(1 - (\mu_i - \mu_j)\alpha^2)$. The best results within 2% of any Standard Model observable are:

Formula	Value	Observable	Error
$f(\mu_2)$	42.699	$1/\alpha_{\text{unified}} \approx 43$	0.70%
$\mu_2/\mu_1 \cdot \log(M_{\text{Pl}}/m_e)$	42.740	$1/\alpha_{\text{unified}} \approx 43$	0.60%
$\mu_2 - \mu_3$	13.113	$\log(v/m_e) = 13.085$	0.21%
$\beta(\mu_0 - \mu_1)/(1 - (\mu_0 - \mu_1)\alpha^2)$	13.365	$\log(1/\sqrt{G_F}/m_e) = 13.259$	0.81%
$\pi/20 \cdot \mu_3/(1 - \mu_3\alpha^2)$	12.155	$\log(M_Z/m_e) = 12.092$	0.52%
$f(\mu_1) - f(\mu_2)$	8.830	$1/\alpha_s(M_Z) = 8.482$	4.11%

Table 4: Best-matching formula variants, sorted by proximity to any known observable.

5 Results

5.1 The Higgs VEV candidate: $\mu_2 - \mu_3$

The most structurally clean near-miss is the bare moment difference:

$$\mu_2 - \mu_3 = \frac{8\pi^3}{21} + \frac{\pi^2}{10} + \frac{\pi}{10} = 13.11303463\dots \quad (17)$$

compared to the logarithm of the Higgs vacuum expectation value:

$$\log\left(\frac{v}{m_e}\right) = \log\left(\frac{246.22 \text{ GeV}}{0.511 \text{ MeV}}\right) = 13.08537\dots \quad (18)$$

where $v = (\sqrt{2}G_F)^{-1/2} = 246.22 \text{ GeV}$ is defined directly from G_F . The fractional error in log-space is **0.21%**, corresponding to $v_{\text{implied}} = e^{\mu_2 - \mu_3} \cdot m_e = 253.1 \text{ GeV}$ versus $v_{\text{measured}} = 246.2 \text{ GeV}$ (2.8% in mass space).

This is the strongest G_F -related proximity result in the moment hierarchy. However, 0.21% is 50 times larger than the 0.004% precision of the P17 gravity formula. The structural question — why the *difference* $\mu_2 - \mu_3$ and not a single moment, and why no self-lensing factor — has no clear answer from the current theory. We state this as an open observation, not a theorem.

5.2 The GUT-scale candidate: $f(\mu_2)$

The P17 formula applied to μ_2 gives:

$$f(\mu_2) = \frac{3\pi}{20} \cdot \frac{\mu_2}{1 - \mu_2\alpha^2} = 42.699. \quad (19)$$

Grand unified theories predict a common coupling at $M_{\text{GUT}} \sim 10^{15}\text{--}10^{16}$ GeV, with $\alpha_{\text{unified}}^{-1} \approx 42\text{--}44$ depending on the model. The standard $SU(5)$ value at the unification scale is approximately $1/\alpha_{\text{unified}} \approx 43.1$. Agreement with $f(\mu_2)$ is 0.70%, which is tantalizing but requires the GUT scale as an external input and falls short of the 0.01% standard.

5.3 Summary table

Moment	Formula	Value	Target	Verdict
μ_0	direct	137.036	$\alpha^{-1} = 137.036$	closed (P3)
μ_1	$f(\mu_1)$	51.530	$\log(M_{\text{P1}}/m_e) = 51.528$	closed (P17), 0.004%
μ_2	$f(\mu_2)$	42.699	$1/\alpha_{\text{unified}} \approx 43$	open, 0.70%
$\mu_2 - \mu_3$	bare diff	13.113	$\log(v/m_e) = 13.085$	open, 0.21%
G_F	any	—	$\log(1/\sqrt{G_F}/m_e) = 13.259$	open
$\alpha_s(M_Z)$	any	—	$1/\alpha_s = 8.482$	open

Table 5: Moment hierarchy: closed results and open problems.

6 Alternative Formula Families

6.1 Factored betas

The P17 coefficient $\beta = 3\pi/20 = \frac{\dim(S^3)\cdot\pi}{\dim(B^4)\cdot V_4}$ carries geometric content. We tested whether μ_2 or μ_3 might connect to the weak or strong interactions via a modified coefficient.

For $\log(M_Z/m_e) = 12.092$, the required coefficient is:

$$\beta_Z = \frac{12.092}{\mu_2/(1 - \mu_2\alpha^2)} = \frac{12.092}{90.611} = 0.1334 = \frac{\pi}{23.54}. \quad (20)$$

This does not correspond to a geometrically motivated integer. The closest natural candidate, $\pi/24 = \dim(S^3)\pi/(8\dim(B^4))$, gives 11.86, off by 0.88%.

For $\alpha_s(M_Z) = 0.1179$, inverting yields $1/\alpha_s = 8.482$. No single-moment formula with a natural coefficient achieves this within 2%. The nearest formula involving μ_n only, $f(\mu_1) - f(\mu_2) = 8.830$, is off by 4.1%.

6.2 The $\mu_0 - \mu_1$ gap

The formula applied to the first moment-gap:

$$\beta \frac{\mu_0 - \mu_1}{1 - (\mu_0 - \mu_1)\alpha^2} = 13.365 \quad (21)$$

is the second-closest candidate for $\log(1/\sqrt{G_F}/m_e) = 13.259$, with 0.81% error. This uses $\mu_0 - \mu_1 = 4\pi^3/5 + \pi^2/4 + \pi/3 = 28.320$, the boundary-to-bulk spectral gap, which may have geometric significance. Not a theorem at current precision.

6.3 Running coupling structure

The $\log(\mu_n/\mu_{n+1})$ sequence shows structure:

$$\log(\mu_0/\mu_1) = 0.231, \quad \log(\mu_1/\mu_2) = 0.186, \quad \log(\mu_2/\mu_3) = 0.157. \quad (22)$$

The decreasing steps are consistent with logarithmic running, suggesting $\mu_n \sim c \cdot e^{-\gamma n}$ for some γ . Whether this logarithmic structure encodes the RG running of couplings is an open structural question.

7 Open Problems

The following are stated precisely for future work:

1. **G_F closure.** Find a formula involving moments μ_n and the coupling α that reproduces $\log(1/\sqrt{G_F}/m_e) = 13.259$ within 0.01%, with a geometrically motivated coefficient. The current best attempt $\mu_2 - \mu_3 = 13.113$ has 0.21% error in log-space (2.8% in v).
2. **α_s closure.** Find a formula reproducing $1/\alpha_s(M_Z) = 8.482$ within 0.01%. This may require identifying the natural scale at which to evaluate α_s (the formula may target $\alpha_s(M_{\text{GUT}})$ rather than $\alpha_s(M_Z)$).
3. **μ_2 physical interpretation.** The P17 conjecture (Conj. 7.1 in [?]) proposes μ_2 encodes the cosmological constant. This should be tested explicitly.
4. **Self-lensing structure for differences.** Why does $\mu_2 - \mu_3$ match $\log(v/m_e)$ bare (without self-lensing), while μ_1 requires the full $f(\mu_1)$ with the $1/(1 - \mu_1\alpha^2)$ factor? A structural explanation is needed.
5. **GUT scale from $f(\mu_2)$.** Is the near-miss $f(\mu_2) \approx 43$ a genuine prediction of the unification scale, or coincidence? A derivation of M_{GUT} from the moment sequence would be required.

8 Conclusion

The moment hierarchy conjecture is not confirmed at theorem precision. The moment structure of the P10/P17 density *does* produce tantalizing proximity results: $f(\mu_2) = 42.70$ is within 0.70% of the GUT-scale unified coupling, and $\mu_2 - \mu_3 = 13.113$ is within 0.21% of $\log(v/m_e)$ where $v = (\sqrt{2}G_F)^{-1/2}$ is the Higgs VEV. Neither reaches the 0.01% bar required for a theorem claim.

The two closed results in the hierarchy remain $\mu_0 = \alpha^{-1}$ (exact) and $f(\mu_1) = \log(M_{\text{Pl}}/m_e)$ at 0.004%. All five attempts to extend the hierarchy to the weak and strong sectors remain open. The systematic record of what was tried and what precision was achieved is documented in Table ?? and in `verify_P242.py` (25/25 checks pass).

References

- [1] L. F. Vlegels, *Gravity as Bulk Propagation: Deriving G from the Dirac Operator on (B^4, S^3)* , This volume, Paper 17 (2025).

- [2] L. F. Vlegels, *The Three-Fourths Boundary Correction: Simplex-Sphere-Hypercube Equilibrium*, This volume, Paper 10 (2025).
- [3] L. F. Vlegels, *Addendum A239: Gravity/Cosmology Tractability Map*, This volume (2026).