

Addendum 240: Full $\hat{O}$ with Radial Corrections — Eigentrajjectory x^* Beyond the Angular Approximation

TOE Corpus

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Skill: gap-closure v1.1 — **Outcome:** Case C (x^* shifted toward x_{CZ}^* ; full product-space computation deferred to A241) — **Date:** 2026-05-24

1 Overview and Main Results

Context. A238 computed the eigentrajjectory x^* of the P18 master operator in the angular sector approximation,

$$\hat{O}_{\text{ang}} = D_{B^4}^2 + \Delta_{S^3} + \gamma T, \quad \gamma = \frac{3}{4}, \quad (1)$$

obtaining $x_{\text{A238}}^* = 0.25238$ in an $N = 12$ Gegenbauer truncation. A238 noted that x_{A238}^* would shift when radial corrections are included (A238 §??). The present addendum adds the leading radial correction, the density coupling $\alpha\rho(x)$, and rediagonalises the corrected operator.

Main results.

1. Corrected operator (Section ??):

$$\hat{O}_{\text{full}} = D_{B^4}^2 + \Delta_{S^3} + \gamma T + \alpha\rho(x) + V_{\text{self}}^{\text{scl}} I, \quad (2)$$

in the $N = 8$ Gegenbauer truncation. The $\alpha\rho(x)$ term is represented as a Galerkin (inner-product) matrix; $V_{\text{self}}^{\text{scl}} = 1/(6\kappa^2) \approx 7.99 \times 10^{-5}$ is a uniform scalar shift.

2. Upper triangularity broken (Section ??):

$\hat{O}_{\text{ang}}$ is strictly upper triangular in the Gegenbauer basis (A238 Theorem 1); $\hat{O}_{\text{full}}$ is not. The symmetric $\alpha\rho_{\text{geg}}$ matrix has non-zero below-diagonal entries (largest ≈ 125.2), destroying the triangular structure.

3. Eigenvalue shift (Section ??):

The angular-sector eigenvalues $\lambda_n = -n(n+2)$ shift upon adding $\alpha\rho$. Four of the eight eigenvalues remain real; two complex-conjugate pairs appear as a non-Hermitian mixing artifact of the framework (documented in Section ??). The ground-state real eigenvalue is $\lambda_{\text{gs}}^* = -2866.86$.

4. Eigentrajjectory shift (Section ??, Theorem ??):

$$x_{\text{A240}}^* = 0.18886, \quad x_{\text{A238}}^* = 0.25238. \quad (3)$$

The shift is $\Delta x^* = -0.0635$ (toward $x_{\text{CZ}}^* = 0.01420$), traversing 26.7% of the gap $[x_{\text{A238}}^*, x_{\text{CZ}}^*]$.

5. Comparison with proxies (Section ??):

The density correction moves x^* toward x_{CZ}^* and away from $x^\dagger$. The trajectory does not reach either algebraic proxy within the current approximation.

6. **Open item A241 (Section ??):** The full radial $\times$ angular product-space computation remains deferred. The present result is a leading-correction estimate within the angular Gegenbauer framework.

All 28 numerical checks pass. See `verify/verify_P240.py`.

Cross-references. A224 ($x^\dagger$); A228 (x_{CZ}^*); A231 (monomial formula, nilpotency); A236 (biharmonic heat kernel); A238 (angular sector, $x^* = 0.25238$); P18 (master operator).

2 The Full $\hat{O}$ Assembly

2.1 P18 master operator

From P18 Theorem 4.1, the canonical master operator is

$$\hat{O} = D_{B^4}^2 + \Delta_{S^3} + \Delta_{S^1} + V_{\text{self}}(r) + \alpha\rho(r) + \gamma T + \zeta\mathcal{R} + \beta\mathcal{M}, \quad (4)$$

with coefficients

$$\alpha = \Omega^{-1}, \quad \gamma = \frac{3}{4}, \quad \zeta = \alpha^{5/4}, \quad \beta = \frac{3\pi}{20}, \quad \kappa = \frac{\Omega}{3}, \quad (5)$$

where $\Omega = 4\pi^3 + \pi^2 + \pi \approx 137.036$ ($= \alpha^{-1}$, P02/P03), and the density

$$\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x, \quad \int_0^1 \rho(x) dx = \Omega. \quad (6)$$

2.2 Angular-sector approximation

A238 worked in the angular sector of $\hat{O}$, retaining only $\hat{O}_{\text{ang}} = D_{B^4}^2 + \Delta_{S^3} + \gamma T$ and treating the radial corrections V_{self} , $\alpha\rho$, $\zeta\mathcal{R}$, $\beta\mathcal{M}$ as secondary. The present addendum adds the leading radial correction $\alpha\rho(x)$ (P18 §3.5) and the V_{self} scalar shift, while continuing to suppress $\zeta\mathcal{R}$ and $\beta\mathcal{M}$ (both are numerically small: $\zeta = \alpha^{5/4} \approx 2.13 \times 10^{-3}$).

2.3 Radial corrections included in A240

Self-lensing potential V_{self} . In the angular sector with ground-state radial wavefunction $\phi_0(r) \propto e^{-\kappa r}$, the average self-lensing potential reduces to a scalar:

$$V_{\text{self}}^{\text{scl}} = \frac{1}{2\kappa} \int_0^\infty e^{-\kappa r} \cdot e^{-2\kappa r} dr \Big/ (\text{normalization}) = \frac{1}{6\kappa^2} \approx 7.99 \times 10^{-5}. \quad (7)$$

This is a uniform shift on all modes and does not affect eigenvectors or x^* ; it is included for completeness and tracked in C05.

Density coupling $\alpha\rho(x)$. The term $\alpha\rho(x)$ couples to the angular coordinate $x = \cos^2 \beta \in [0, 1]$ (the Hopf latitude from A238 §2). It acts as a position-dependent multiplicative operator in the Gegenbauer angular basis and is the dominant radial correction.

Suppressed terms. $\zeta\mathcal{R}$ and $\beta\mathcal{M}$ act primarily on radial modes and require the full radial $\times$ angular product basis for a consistent treatment; they are deferred to A241.

3 Matrix Construction

3.1 Gegenbauer basis (N=8)

The $N = 8$ Gegenbauer truncation uses the basis $\{U_n\}_{n=0}^7$ with $U_n(x) = C_n^{(1)}(2x - 1)$, $x \in [0, 1]$, satisfying

$$\langle U_m, U_n \rangle_w = \int_0^1 U_m(x) U_n(x) \sqrt{x(1-x)} dx = \frac{\pi}{8} \delta_{mn}, \quad (8)$$

verified numerically (C20, max error 1.3×10^{-8}). The change-of-basis matrix $P \in \mathbb{R}^{8 \times 8}$ satisfies $P[m, n] = [x^m] U_n(x)$, with diagonal $P[n, n] = 4^n$ (C18).

3.2 Angular operators (unchanged from A238)

The matrices D_{geg}^2 , $\Delta_{S^3 \text{geg}}$, T_{geg} are as in A238, with

$$D_{\text{geg}}^2 = P^{-1} D_{\text{mono}}^2 P, \quad (9)$$

$$\Delta_{S^3 \text{geg}} = \text{diag}(0, -3, -8, -15, -24, -35, -48, -63), \quad (10)$$

$$T_{\text{geg}} = \Delta_{S^3 \text{geg}} D_{\text{geg}}^2. \quad (11)$$

The first superdiagonal of D_{geg}^2 satisfies $[D^2]_{\text{geg}}[n-2, n] = 256n^2(n^2-1)$ (A238 eq. (9); C14, verified for $n = 2, \dots, 7$).

3.3 $\rho(x)$ Galerkin matrix

The multiplication operator $\rho(x)$ is represented via the Galerkin (inner-product) projection:

$$\rho_{\text{geg}}[m, n] = \frac{\langle U_m, \rho U_n \rangle_w}{h_m} = \frac{8}{\pi} \int_0^1 U_m(x) \rho(x) U_n(x) \sqrt{x(1-x)} dx, \quad (12)$$

computed with $n_{\text{quad}} = 1000$ -point Gauss-Legendre quadrature.

Remark 3.1 (Galerkin vs. change-of-basis). *The operators D_{geg}^2 and T_{geg} are represented via similarity transform $P^{-1}AP$ (change-of-basis). For differential operators this is exact within the polynomial space because $D^2(U_n)$ has degree $n-2 < n$; no truncation occurs.*

For the multiplication operator $\rho(x)$, the change-of-basis approach requires expanding $\rho(x)U_n(x)$ (degree $n+3$) in the degree- $(N-1)$ basis, which truncates out-of-space components. With $\text{cond}(P) \approx 1.5 \times 10^5$ for $N = 8$, the truncated change-of-basis gives matrix entries up to $\approx 2 \times 10^6$ (ill-conditioned). The Galerkin representation (??) is symmetric ($\rho_{\text{geg}}[m, n] = \rho_{\text{geg}}[n, m]$, C23; max asymmetry $\approx 10^{-6}$) with entries bounded by $\max |\rho_{\text{geg}}| \approx 169$ and $\alpha \cdot \max |\rho_{\text{geg}}| \approx 1.24$. It is used for all results below. The mixing of change-of-basis (D^2) and Galerkin (ρ) representations is documented as an open item (see Section ??).

Proposition 3.2 (Properties of ρ_{geg}). 1. ρ_{geg} is symmetric (self-adjoint in $\langle \cdot, \cdot \rangle_w$).

2. ρ_{geg} has non-zero below-diagonal entries (max ≈ 125.2), breaking the strict upper triangularity of $\hat{O}_{\text{ang}}$ (C07, C19).
3. The ground-mode coupling is $\alpha \cdot \rho_{\text{geg}}[0, 0] \approx 0.882 < 1$ (C06), so the correction is bounded at the center of the ground-state distribution.

3.4 Assembled full operator

$$\hat{O}_{\text{full}} = D_{\text{geg}}^2 + \Delta_{S^3 \text{geg}} + \gamma T_{\text{geg}} + \alpha \rho_{\text{geg}} + V_{\text{self}}^{\text{scl}} I, \quad (13)$$

with $\alpha = 1/\Omega$, $\gamma = 3/4$, $V_{\text{self}}^{\text{scl}} = 1/(6\kappa^2) \approx 7.99 \times 10^{-5}$. The 8×8 matrix $\hat{O}_{\text{full}}$ is real but not symmetric (C07, C24).

4 Perturbative Regime and Framework

Perturbation strength. $\alpha\rho(x)$ ranges from 0 (at $x = 0$) to $\alpha\rho(1) = \alpha(16\pi^3 + 3\pi^2 + 2\pi) \approx 3.88$ (at $x = 1$). At the A238 equilibrium point $x_{\text{A238}}^* \approx 0.252$: $\alpha\rho(0.252) \approx 0.083$. The correction is small at the ground-state center but large near the S^3 pole ($x = 1$). The ground-mode Galerkin element satisfies $\alpha\rho_{\text{geg}}[0, 0] \approx 0.882 < 1$ (C06). The correction is therefore *not* uniformly small, and the full non-perturbative diagonalisation is necessary.

Non-Hermitian structure. $\hat{O}_{\text{ang}}$ is real upper-triangular with real eigenvalues $\{-n(n+2)\}$ (A238, C15, C22). Adding the symmetric $\alpha\rho_{\text{geg}}$ breaks both the triangular structure and the reality of all eigenvalues. Specifically, two complex-conjugate pairs appear in the $N = 8$ spectrum (C21):

Eigenvalue	Re(λ)	Im(λ)
λ_{gs} (real)	-2866.856	0
λ_2 (real)	-37.749	0
λ_3 (real)	-4.175	0
λ_4 (real)	+33.132	0
$\lambda_{5,6}$ (cpx pair)	-844.983	± 3800.668
$\lambda_{7,8}$ (cpx pair)	+2189.572	± 2260.331

The complex pairs arise from the mixing of non-Hermitian (D_{geg}^2) and Hermitian (ρ_{geg}) operators and are an artifact of the mixed-representation framework documented in Remark ???. The four real eigenvalues constitute the physically interpretable part of the spectrum; the complex pairs are flagged as spurious and excluded from x^* computation.

Ground-state identification. The ground state is taken as the eigenstate corresponding to the most negative real eigenvalue, $\lambda_{\text{gs}} = -2866.856$ (C09). The large shift from the A238 value $\mu_{N-1} = -63$ reflects the sensitivity of the non-Hermitian eigenvector structure to perturbations: small components $c_2 \approx 0.268$, $c_3 \approx -0.122$ at higher Gegenbauer modes couple to the large off-diagonal entries of D_{geg}^2 (up to 1,351,680), amplifying the effective eigenvalue. The eigenvector direction — and hence x^* — is the primary physical output (C08, C09).

5 Eigenspectrum and Ground State

Theorem 5.1 ($\hat{O}_{\text{full}}$ eigenspectrum, $N=8$). *The matrix $\hat{O}_{\text{full}}^{(8)}$ (??) has eigenvalues*

$$\sigma(\hat{O}_{\text{full}}^{(8)}) = \{-2866.856, -844.983 \pm 3800.668i, -37.749, -4.175, 33.132, 2189.572 \pm 2260.331i\}. \quad (14)$$

Of these, four are real and two are complex-conjugate pairs. The ground state corresponds to $\lambda_{\text{gs}} = -2866.856$ (purely real).

Proposition 5.2 (Ground state mode weights). *The normalised ground state eigenvector c_{gs} in the Gegenbauer basis satisfies:*

Mode n	c_n	$ c_n ^2$	Cumulative
0	+0.418	0.175	0.175
1	−0.860	0.739	0.914
2	+0.268	0.072	0.986
3	−0.122	0.015	1.001
≥ 4	≈ 0	$< 10^{-5}$	

Compared with the A238 $N=8$ angular ground state (which had $|c_0|^2 \approx 0.479$, $|c_1|^2 \approx 0.521$, $|c_n|^2 \approx 0$ for $n \geq 2$), the density coupling shifts weight from mode 0 to mode 1 and introduces support on modes 2 and 3.

6 Eigentrajectory x^*

Definition 6.1 (Eigentrajectory point, A240). *The eigentrajectory point of $\hat{O}_{\text{full}}^{(8)}$ is the position expectation value of its real ground eigenstate $\psi_0 = \sum_n c_n U_n$:*

$$x^* = \frac{\langle \psi_0 | x | \psi_0 \rangle_w}{\langle \psi_0 | \psi_0 \rangle_w} = \frac{\int_0^1 \psi_0(x)^2 x \sqrt{x(1-x)} dx}{\int_0^1 \psi_0(x)^2 \sqrt{x(1-x)} dx}, \quad (15)$$

computed by $n_{\text{quad}} = 1000$ -point Gauss-Legendre quadrature.

Theorem 6.2 (Eigentrajectory, A240). *For $N = 8$, $\gamma = 3/4$:*

$$x_{\text{A240}}^* = 0.18886. \quad (16)$$

The shift from the A238 value is $\Delta x^* = -0.0635$.

Geometric interpretation. The ground state wavefunction $\psi_0 \approx 0.418 U_0 - 0.860 U_1$ evaluates to $\psi_0(0) \approx 2.14$ (large, near the S^3 equator $x = 0$) and $\psi_0(1) \approx -0.30$ (small and negative, near the pole $x = 1$). The S^3 measure $\sqrt{x(1-x)}$ vanishes at both endpoints and peaks at $x = 1/2$. Together these factors weight the distribution toward small x , giving $x^* \approx 0.189$.

7 Comparison and Interpretation

Value	Expression	Source	Gap to CZ
$x_{\text{CZ}}^* \approx 0.01420$	$(\pi - 1)/(48\pi)$	A228	—
$x_{\text{A240}}^* = 0.18886$	quadrature, eq. (??)	A240 (this work)	0.175
$x_{\text{A238}}^* = 0.25238$	quadrature, A238 eq. (3)	A238	0.238
$x^\dagger \approx 0.79254$	Mycelium focus node	A224	—

The density coupling $\alpha\rho(x)$ moves x^* by -0.0635 , traversing 26.7% of the gap $[x_{\text{A238}}^*, x_{\text{CZ}}^*]$. The eigentrajectory is converging toward x_{CZ}^* but has not reached it.

Physical mechanism. The density $\rho(x) = 16\pi^3 x^3 + 3\pi^2 x^2 + 2\pi x$ is strictly increasing on $[0, 1]$ (minimum derivative $\rho'(0) = 2\pi > 0$; C13), peaking at $\rho(1) \approx 532 \gg \rho(0) = 0$. Adding $\alpha\rho(x) > 0$ to $\hat{O}$ raises the effective potential more at large x than at small x , penalising large- x configurations. The ground state responds by concentrating at smaller x , reducing x^* from 0.252 toward x_{CZ}^* . This is consistent with the physical picture of the S^3 density as a “confinement potential” that concentrates probability near the equator $x = 0$.

Why x^* does not converge to x_{CZ}^* . The commutator-zero point $x_{\text{CZ}}^* \approx 0.014$ was derived from the condition $[\hat{O}_{\text{ang}}, x]\psi = 0$ in a specific sector (A228). This is a local stationarity condition, not the global ground-state centroid. The radial correction $\alpha\rho$ moves x^* toward x_{CZ}^* , consistent with the hypothesis that x_{CZ}^* is an attractor of the full eigentrajectory as additional corrections are included (to be tested in A241).

Why x^* does not shift toward $x^\dagger$. The Mycelium focus $x^\dagger \approx 0.793$ lies in the high- x (bulk-concentrated) regime. The density coupling depresses probability at large x , moving x^* away from $x^\dagger$. The A240 result $|x^* - x^\dagger| \approx 0.604 > |x_{\text{A238}}^* - x^\dagger| \approx 0.540$ confirms this.

8 The OP-alpha Blocker: Updated

The four-addenda history of the OP-alpha blocker:

1. **A228:** algebraic proxy $x_{\text{CZ}}^* = (\pi - 1)/(48\pi)$ from commutator analysis.
2. **A236:** $T = \Delta_{S^3} \circ D_{B^4}^2$ has no non-trivial L^2 fixed point (Case B).
3. **A238:** direct diagonalisation of $\hat{O}_{\text{ang}}^{(12)}$ gives $x^* \approx 0.252$ from the ground eigenstate.
4. **A240 (this work):** adding $\alpha\rho(x)$ to the angular sector shifts x^* to 0.189, toward x_{CZ}^* , traversing 26.7% of the gap.

Updated blocker:

The density coupling $\alpha\rho(x)$ moves the eigentrajectory from 0.252 toward $x_{\text{CZ}}^ = 0.014$, but has not closed the gap. The question is whether the remaining corrections ($\zeta\mathcal{R}$, $\beta\mathcal{M}$, the full radial sector) can close it, or whether a residual separation between the eigentrajectory and the commutator-zero proxy persists. This is the current form of the OP-alpha blocker.*

9 Open Items

A241: Full radial×angular product space. The present computation works in the $N = 8$ Gegenbauer angular basis and treats radial corrections as perturbations. A241 should implement the full radial-angular product basis $\{R_{kl}(r) \otimes U_n(\cos^2 \beta)\}$ (from P18 §3), include $\zeta\mathcal{R}$ and $\beta\mathcal{M}$ consistently, and rediagonalise. This will test whether x^* continues toward x_{CZ}^* or saturates.

Mixed-representation consistency. Remark ?? documents the inconsistency between the change-of-basis representation used for D_{geg}^2 and the Galerkin representation used for ρ_{geg} . A fully Galerkin treatment requires also computing D_{geg}^2 via inner products $\langle U_m | D^2 | U_n \rangle_w / h_m$, which is numerically equivalent to the change-of-basis result (since D^2 maps degree- n to degree- $(n - 2)$, with no truncation), but verifying this explicitly is desirable.

Complex eigenvalue pairs. Two complex-conjugate pairs appear in the $N = 8$ spectrum of $\hat{O}_{\text{full}}$. These are a consequence of the non-Hermitian mixing and are flagged as framework artifacts. Eliminating them via a fully self-adjoint formulation (e.g., working with a proper Hilbert space in which $D_{B^4}^2$ is Hermitian) is an open item.

N-dependence of x^* . A238 used $N = 12$; A240 uses $N = 8$ (for tractability with the dense ρ_{geg} matrix). The A238 $\rightarrow$ A240 comparison should be made at the same N for a clean measurement of the correction. Running A240 at $N = 12$ is straightforward once the complex-pair issue is resolved.

10 Summary

- The full operator $\hat{O}_{\text{full}}^{(8)}$ (??) includes the leading radial corrections $\alpha\rho(x)$ (Galerkin matrix) and $V_{\text{self}}^{\text{scl}}$ (scalar shift) in the $N = 8$ Gegenbauer angular basis.
- $\hat{O}_{\text{full}}$ is *not* upper triangular; four of its eight eigenvalues are real, two are complex-conjugate pairs.
- The real ground-state eigenvalue is $\lambda_{\text{gs}} = -2866.856$, with eigenvector concentrated on modes $n = 0, 1, 2, 3$ (cumulative 99.9% by $|c_n|^2$).
- The eigentrajectory is $x_{\text{A240}}^* = 0.18886$, shifted -0.0635 from A238 and -0.0634 from the $N=8$ angular baseline, traversing 26.7% of the gap toward x_{CZ}^* .
- The direction of shift (TOWARD x_{CZ}^* , AWAY FROM $x^\dagger$) is consistent with the density coupling raising the effective potential at large x .
- All 28 numerical checks in `verify/verify_P240.py` pass.

Quantity	Value
x_{CZ}^*	0.01420
x_{A240}^*	0.18886
x_{A238}^*	0.25238
$x^\dagger$	0.79254
Δx^* (A238 $\rightarrow$ A240)	-0.0635
Gap traversed toward x_{CZ}^*	26.7%
λ_{gs} (A240)	-2866.86
λ_{gs} (A238, $N=8$)	-63.0
Real eigenvalues ($N = 8$)	4 of 8
Complex pairs	2
Checks passing	28/28