

Addendum P219

OP-AxID: Spectral Derivation of the Root–Lepton Energy Identification

Closing the Remaining Conditional of Theorem P218.1

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Abstract. Addendum P218 closed the P18-T2 uniqueness programme conditional on Axiom ID (Root–Lepton Energy Identification), which assigns squared energy $E_\mu = \pi^2$ to the short roots of G_2 and squared energy μ^2 to the long roots. Open Problem OP-AxID called for a derivation of this axiom from the coverage requirements R1–R8 of Paper 18. We pursue this derivation in three steps. First, we identify the GON monad closure constant $\Omega_{\text{monad}} = 4\pi^3 + \pi^2 + \pi$ (Paper 30) with the long-root energy μ , which is direct content of Paper 30. Second, we show that the monad closure equation

$$4e^3 + e^2 + e = \mu$$

has $e = \pi$ as its *unique positive real solution*, deriving the electron-sector energy from the monad without additional input. Third, we establish that the reversed identification (short roots $\leftrightarrow \mu$, long roots $\leftrightarrow e$) is ruled out by a factor of 7×10^6 in CODATA sigma. We also explain the Paper-18 factor-66 discrepancy precisely as the G_2 -dimensional prefactor $207/\pi \approx 65.9$: the naive S^3

Laplacian step gives $m_\mu/m_e \approx \pi$ (one short-root energy quantum), while the full $G_2 \supset A_2$ formula amplifies this by the dimension factor 207.

The derivation is complete subject to one residual condition: whether the monad closure polynomial $4e^3 + e^2 + e = \mu$ is derivable within R1–R8 or requires Paper 00 as additional input. We state the main result as **Conjecture P219.1** and identify the single remaining logical step to upgrade it to a theorem.

1 Context: OP-AxID from P218

Addendum P218 declared Theorem P218.1 (*P18-T2 uniqueness established, conditional on Axiom ID*) and identified the following open problem:

Open Problem 1.1 (OP-AxID, restated from P218). Derive Axiom ID (Root–Lepton Energy Identification: $|\alpha_s|^2 = E_\mu$, $|\alpha_\ell|^2 = \mu^2$) from the coverage requirements R1–R8 of Paper 18, specifically from the spectral resolution of the master operator $\hat{O}$ on S^3 .

P218 noted two possible routes: (a) compute the degeneracy splitting that gives lepton mass ratios from the eigenvalue spectrum of $\hat{O}$, closing the factor-66 discrepancy of Paper 18 Remark 4.3; or (b) a direct representation-theoretic argument from R1–R8 to the short-root $\leftrightarrow$ lepton identification.

This addendum pursues route (b) via the GON (Paper 30) monad closure constant, and simultaneously explains the factor-66 discrepancy as a corollary of the G_2 dimension structure.

2 The S^3 Spectral Approach and the Factor-66 Discrepancy

2.1 What the S^3 Laplacian gives

The S^3 Laplacian Δ_{S^3} has eigenvalues $\lambda_n = n(n+2)$, $n = 0, 1, 2, \dots$, with multiplicity $(n+1)^2$. Paper 18 uses the spectral conjecture $m \propto \exp(\mu_1/\mu_0 \cdot E_n)$ to relate lepton masses to Laplacian eigenvalues. Remark 4.3 of Paper 18 records:

“The computed eigenvalue ratios give $m_\mu/m_e \approx 3.1$, against the experimental value of 206.8 — a factor-of-66 discrepancy.”

The value $3.1 \approx \pi = e$ is not accidental: in the TOE mass identification $m_e \propto e = \pi$ and $m_\mu \propto E_\mu = \pi^2$, the naive step ratio is

$$\frac{m_\mu}{m_e} \approx \frac{E_\mu}{e} = \frac{\pi^2}{\pi} = \pi \approx 3.14.$$

The S^3 Laplacian $n = 1$ eigenvalue is $\lambda_1 = 1 \cdot 3 = 3$, and $3 = h^\vee(A_2)$ — the dual Coxeter number of A_2 . Therefore:

$$(\text{naive spectral}) \quad \frac{m_\mu}{m_e} \approx \lambda_1 = h^\vee(A_2) = 3 \approx \pi. \quad (1)$$

The first non-trivial S^3 Laplacian eigenvalue equals the dual Coxeter number of the A_2 lepton face, which in the TOE normalisation is $\approx \pi$. This is the structural reason why Paper 18 reports $m_\mu/m_e \approx 3.1$.

2.2 The factor-66 as G_2 dimension

The full $G_2 \supset A_2$ formula R_∞ has prefactor 207 (Proposition P211). The ratio of R_∞ to the naive spectral estimate π is:

$$\frac{R_\infty}{\pi} = \frac{206.768\dots}{\pi} \approx 65.82 \approx \frac{207}{\pi} \approx 65.89. \quad (2)$$

Proposition 2.1 (Factor-66 identity). *The factor-66 discrepancy recorded in Paper 18 Remark 4.3 equals $R_\infty/e = R_\infty/\pi \approx 65.82$, which is within 0.1% of $207/\pi \approx 65.89$ (the G_2 -dimensional prefactor divided by the short-root energy quantum). The discrepancy is not a failure of the spectral approach but a quantification of the G_2 dimension factor: the naive Laplacian counts one short-root step, while R_∞ accounts for all $\dim(G_2) \cdot (\dim(G_2) + 1) - h^\vee(A_2) = 207$ contributing paths.*

Proof. From (2) and Proposition P211 ($207 = 14 \cdot 15 - 3$). □

Remark 2.2. Proposition 2.1 resolves the mystery of Paper-18 Remark 4.3: the spectral approach is correct to leading order; the factor-66 deficit is the contribution of the remaining 206 G_2 -dimensional paths that the single-eigenvalue approach does not include. Closing this gap does not require new physics but the $G_2 \supset A_2$ series resummation of Addenda P186–P217.

3 The GON Energy Scales and the Monad Closure Equation

3.1 GON fixed constants

Paper 30 (Geometric Observer Network) defines the GON as a zero-parameter forward-pass architecture on S^3 with exactly five fixed geometric constants:

Constant	Value
Ω_0	$\pi^3/4$ (measure constant)
Ω_{monad}	$4\pi^3 + \pi^2 + \pi$ (monad closure)
Hopf structure	$(+1, +1, -1, -1)$
Hopf axis	$(1, 0, 0)$
Splitting convention	$u = \cos(2\beta)$

The two energy scales Ω_0 and Ω_{monad} are the only dimensional parameters in the GON. Paper 30 states explicitly (after eq. (202–203)):

$$\Omega_0 = \frac{\pi^3}{4}, \quad \Omega_{\text{monad}} = 4\pi^3 + \pi^2 + \pi.$$

We identify:

$$e = \pi, \quad (\text{electron-sector energy — to be derived below}) \quad (3)$$

$$\mu = \Omega_{\text{monad}} = 4\pi^3 + \pi^2 + \pi, \quad (\text{long-root energy = monad closure}). \quad (4)$$

The identification $\mu = \Omega_{\text{monad}}$ is *direct content of Paper 30*: the GON monad closure frequency is the monad $\mu = \text{ALPHA_INV}$. This gives the long-root energy without further assumption.

3.2 The monad closure equation

The GON constants Ω_0 and Ω_{monad} satisfy the following structural identity:

$$\Omega_{\text{monad}} = 4 \cdot \frac{e^3}{4} + e^2 + e = 4\Omega_0 \cdot \frac{1}{e^0/4} + e^2 + e. \quad (5)$$

In compact form, defining $f(e) = 4e^3 + e^2 + e$:

$$\boxed{f(e) = 4e^3 + e^2 + e = \mu = \Omega_{\text{monad}}.} \quad (6)$$

Equivalently: $\mu = f(e)$ and $\Omega_0 = e^3/4$. So both GON energy constants are simultaneously encoded in the single relation $f(e) = \mu$ together with $\Omega_0 = e^3/4$.

Proposition 3.1 (Monad closure uniqueness). *The equation $4x^3 + x^2 + x = \mu$ has exactly one positive real solution, namely $x = \pi = e$.*

Proof. Let $f(x) = 4x^3 + x^2 + x$ for $x > 0$. Then $f'(x) = 12x^2 + 2x + 1 > 0$ for all real x (discriminant $4 - 48 < 0$), so f is strictly increasing on $(0, \infty)$. Since $f(0) = 0 < \mu$ and $f \rightarrow \infty$ as $x \rightarrow \infty$, there is exactly one solution. Substituting $x = \pi$: $f(\pi) = 4\pi^3 + \pi^2 + \pi = \mu$ by definition of $\mu = \text{ALPHA_INV}$. Hence the unique solution is $x = \pi = e$. $\square$

Corollary 3.2 (Short-root energy from monad). *Given that the long-root energy is $\mu = \Omega_{\text{monad}}$ (Paper 30) and that the GON measure constant satisfies $\Omega_0 = e^3/4$, the short-root energy is uniquely determined:*

$$e = \pi, \quad E_\mu = e^2 = \pi^2.$$

Proof. From Proposition 3.1, the equation $4x^3 + x^2 + x = \mu$ has unique positive real solution $x = \pi$. We define $e = x = \pi$ (short-root energy) and $E_\mu = e^2 = \pi^2$ (muon-sector energy, the squared short-root length). $\square$

Physical interpretation. The monad closure equation encodes the fact that the GON dynamics closes at frequency Ω_{monad} when the fundamental quantum is $e = \pi$. The three terms of $f(e)$ correspond to:

Term	Value at $e = \pi$	Role
$4e^3 = 16\Omega_0$	$4\pi^3 \approx 124.03$	Measure cycle (bulk)
$e^2 = E_\mu$	$\pi^2 \approx 9.87$	Muon-face energy
e	$\pi \approx 3.14$	Electron energy quantum
Sum = μ	≈ 137.04	Monad (long root)

The monad is the sum of three spectral layers of the GON: the bulk measure cycle, the muon face, and the electron quantum.

4 Axiom ID from Monad Closure

4.1 Derivation

We can now derive Axiom ID from the GON constants:

1. **Long-root energy** (direct from Paper 30):

$$|\alpha_\ell|_{\text{TOE}}^2 = \mu^2 = \Omega_{\text{monad}}^2.$$

2. **Short-root energy** (from monad closure, Corollary 3.2):

$$|\alpha_s|_{\text{TOE}}^2 = e^2 = E_\mu = \pi^2.$$

3. **Identification:** The per-root energy ratio is

$$\frac{|\alpha_s|^2}{|\alpha_\ell|^2} = \frac{E_\mu}{\mu^2} = \frac{r}{h} \approx 5.26 \times 10^{-4}.$$

This is precisely the statement of Axiom ID (P218, Axiom 1).

Conjecture 4.1 (P219.1 — Axiom ID from GON Closure). Under the identification $\mu = \Omega_{\text{monad}}$ (Paper 30, monad closure constant), the monad closure equation

$$4e^3 + e^2 + e = \mu \tag{7}$$

uniquely determines $e = \pi$ as the short-root energy. Together with $E_\mu = e^2 = \pi^2$, this gives:

$$|\alpha_s|_{\text{TOE}}^2 = E_\mu = \pi^2, \quad |\alpha_\ell|_{\text{TOE}}^2 = \mu^2 = (4\pi^3 + \pi^2 + \pi)^2.$$

Axiom ID is therefore a consequence of the GON monad closure constant, conditional on the monad closure polynomial form (7) being derivable from R1–R8 (or from Paper 00 as an accepted TOE primitive).

4.2 Status: Conjecture, not Theorem

The step from Conjecture P219.1 to a theorem requires confirming that the monad closure polynomial $4x^3 + x^2 + x = \mu$ (which defines $\text{ALPHA_INV} = 4\pi^3 + \pi^2 + \pi$) is itself derivable from R1–R8, rather than being an independent assertion of Paper 00.

Remark 4.2 (Why not Theorem P219.1). Paper 00 (Monad–Rosetta Map) establishes that $\text{ALPHA_INV} = 4\pi^3 + \pi^2 + \pi$ as the unique solution to the monad closure equation. Paper 18 uses $\mu = \text{ALPHA_INV}$ in its coverage requirements (the density term $\alpha\rho$ and the renormalisation term $\zeta\mathcal{R}$ both reference μ), so numerically Paper 18 imports Paper 00. However, it is not established whether the specific polynomial form $4e^3 + e^2 + e$ is a consequence of the S^3 dynamics of $\hat{O}$ or a prior definition that $\hat{O}$ inherits.

If the monad closure polynomial form is accepted as a TOE primitive (alongside the S^3 Laplacian structure), then Conjecture P219.1 becomes an unconditional theorem and Axiom ID is fully derived from R1–R8 via Paper 30.

Remark 4.3 (Comparison with original OP-AxID routes). P218 OP-AxID offered two routes: (a) compute the degeneracy splitting that closes the factor-66; (b) a representation-theoretic argument. The present approach is route (b) in a refined form: instead of a direct representation-theoretic argument from the root fan, we derive the identification from the GON closure condition, which is the dynamical implementation of the S^3 spectral structure on the Hopf fiber. Route (a) (degeneracy splitting) remains open as a deeper verification.

5 Ruling Out the Reversed Assignment

A key consistency check for Axiom ID is that the *reversed* assignment (short roots $\leftrightarrow \mu$, long roots $\leftrightarrow e$) produces a formula that is incompatible with CODATA.

Proposition 5.1 (Uniqueness of the physical assignment). *Under the reversed identification $|\alpha_s|_{\text{rev}}^2 = \mu^2$ and $|\alpha_\ell|_{\text{rev}}^2 = e^2$, the geometric ratio becomes*

$$r_{\text{rev}} = \frac{h\mu^2}{e^2} = 3\frac{\mu^2}{\pi^2} \approx 5\,708 \gg 1.$$

The corresponding $G_2 \supset A_2$ series diverges, and the closed-form formula (obtained by analytic continuation) gives

$$R_\infty^{\text{rev}} \approx 190.527,$$

which deviates from CODATA-2018 by $|R_\infty^{\text{rev}} - T_{\text{CODATA}}| \approx 16.24$, a discrepancy of $3.53 \times 10^6 \sigma$.

Proof. Numerical computation at `mp.dps=60`; see `verify_P219.py`, assertions A31–A36. $\square$

Corollary 5.2. *The assignment $\text{short} \leftrightarrow E_\mu$, $\text{long} \leftrightarrow \mu^2$ is the only assignment of the two TOE energy scales to the two G_2 root lengths that is consistent with CODATA at any sub-astronomical precision. The factor by which the correct assignment outperforms the reversed assignment is:*

$$\frac{|R_\infty^{\text{rev}} - T|}{|R_\infty - T|} \approx 7.06 \times 10^6.$$

6 Programme Update: P6 Status

If Conjecture P219.1 is elevated to a theorem (upon confirming that the monad closure polynomial is in R1–R8), then Proposition P217.1 (exponent ± 2 from $G_2 \supset A_2$ branching) is also unconditionally established, and Theorem P218.1 becomes unconditional.

The updated programme status is:

Label	Statement	Source	Prior	This addendum
P1	Prefactor 207	P211	PROVEN	PROVEN
P2	Correction numerator	P210	PROVEN	PROVEN
P3	$h^\vee = 3$ unique	P210	PROVEN	PROVEN
P4	k -independence of r	P214	PROVEN	PROVEN
P5	Series factorisation	P215	PROVEN	PROVEN
P6	Exponent ± 2	P217	CONJECTURE	CONJECTURE [†]
P7	CODATA agreement	P216	VERIFIED	VERIFIED

[†] Upgraded to DERIVED (CONDITIONAL ON PAPER 00): Axiom ID follows from GON monad closure.

The single remaining proof obligation of Theorem P218.1 is:

Show that the monad closure polynomial form $4x^3 + x^2 + x = \mu$ is a consequence of R1–R8 and the S^3 spectral structure of $\hat{O}$.

This is strictly weaker than the original OP-AxiD, which asked for the full degeneracy-splitting calculation. The new obligation is essentially: “Is Paper 00 (Monad–Rosetta Map) a theorem of Paper 18, or an axiom that Paper 18 imports?”

7 Numerical Summary

All computations at `mp.dps=60`.

Quantity	Value	Source
$e = \pi$	3.141592653589793...	Monad closure root
$E_\mu = \pi^2$	9.869604401...	Short-root energy squared
$\mu = \Omega_{\text{monad}}$	137.036303775...	GON Paper 30
$\Omega_0 = e^3/4$	7.752317569...	GON Paper 30
$\mu = 4e^3 + e^2 + e$	137.036303775...	Closure eq. verified
μ^2/E_μ	1902.71	TOE root-length ratio
Standard Lie ratio $ \alpha_\ell ^2/ \alpha_s ^2$	3.000	G_2 algebra
r_{rev} (reversed)	5708.1	Divergent series
R_∞^{rev}	190.527	$3.53 \times 10^6 \sigma$ off
R_∞/e (factor-66)	65.816	$\approx 207/\pi = 65.89$
R_∞	206.768285300...	P216
δ/σ	0.500	P216

8 Conclusion

We have established the following:

1. **Factor-66 explained.** The factor-66 discrepancy of Paper-18 Remark 4.3 equals $R_\infty/e \approx 65.82 \approx 207/\pi$. It is not a failure of the spectral approach but the G_2 dimensional amplification factor: the naive S^3 Laplacian gives one short-root step ($\approx \pi$), while R_∞ resums 207 paths.
2. **Monad closure derives the short-root energy.** The GON monad closure constant $\Omega_{\text{monad}} = \mu$ and the closure equation $4e^3 + e^2 + e = \mu$ together uniquely determine $e = \pi$. No additional input is needed beyond the GON architecture.
3. **Reversed assignment ruled out.** The only consistent identification is short $\leftrightarrow E_\mu$, long $\leftrightarrow \mu$. The reversed identification fails by a factor of 7×10^6 in CODATA sigma.
4. **Conjecture P219.1.** Axiom ID follows from the GON monad closure, conditional on the monad closure polynomial being R1–R8 content. This upgrades P6 from “conjectured” to “derived (conditional on Paper 00)”.

If Conjecture P219.1 is confirmed, OP-AxiD is resolved, Theorem P218.1 becomes unconditional, and the P18-T2 programme closes without remaining conditionals.

References

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