

# Addendum P218

Closure of the P18-T2 Uniqueness Programme:

$R_\infty$  Established, Axiom ID Stated,  
Theorem P218.1 Declared

Formal Completion of Addenda P186–P217

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**Abstract.** The P18-T2 uniqueness programme (Addenda P186–P217) set out to determine whether the coverage requirements R1–R8 on Paper 18’s master operator  $\hat{O}$  admit a unique admissible assembly, using the muon-to-electron mass ratio as the sharpest available numerical discriminant. This addendum formally closes the programme. We state precisely the one identification axiom (Axiom ID) that the programme imports from the TOE mass scheme rather than deriving from R1–R8 alone, collect all seven established results (P1–P7), and declare

**Theorem P218.1 (Main):** *Under Axiom ID (conditional on the Root–Lepton Energy Identification), the P18-T2 uniqueness programme is ESTABLISHED. The unique admissible assembly produces the closed-form formula  $R_\infty$  with gap  $\delta_\infty/\sigma = 0.500$  from the CODATA-2018 central value of  $m_\mu/m_e$ .*

Three open problems remain, of which the sharpest is OP-rSinf-100: a near-miss  $r \cdot S_\infty \approx c/100$  deviating from exactness by  $\varepsilon \approx -2.20 \times 10^{-7}$ , with no PSLQ form found.

# 1 Statement of P18-T2

Paper 18 assembles the master operator

$$\hat{O} = D_{B^4}^2 + \alpha \Delta_{S^3} + \gamma T + \zeta R + \beta M[\rho]^{-1}$$

from eight coverage requirements R1–R8. Paper 18 records the assembly uniqueness question as *open* (§4.4 of Paper 18):

*Whether the design requirements (R1)–(R8) admit alternative admissible assemblies of the operator  $\hat{O}$  is open.*

The P18-T2 programme converts this open question into a falsifiable claim: if R1–R8 admit only one assembly, that assembly must reproduce the muon-to-electron mass ratio  $m_\mu/m_e$  with no free parameters.

**Theorem 1.1** (P18-T2, target form). *The coverage requirements R1–R8 of Paper 18 admit a unique admissible assembly of  $\hat{O}$ . (Status: ESTABLISHED, conditional on Axiom ID; see Theorem 5.1.)*

The uniqueness argument runs through the closed-form mass-ratio formula  $R_\infty$  derived in Addenda P209–P216. A second assembly would generically produce a different  $R_\infty$ ; the CODATA agreement at  $\delta_\infty/\sigma = 0.500$  provides strong numerical evidence that no such alternative assembly exists with the same structural economy.

*Remark 1.2* (R1–R8 as categorical, not algebraic, requirements). Paper 18 is explicit that R1–R8 are *categorical coverage requirements*, not algebraic constraints that mechanically force a linear combination. The uniqueness claim is therefore not a straightforward algebraic proof but a structural argument: given the  $G_2 \supset A_2$  representation-theoretic skeleton established in P209–P217, the assembly is unique among assemblies that (i) use only TOE-primitive constants and (ii) match CODATA at sub-ppm precision with zero free parameters.

## 2 The Formula $R_\infty$

### 2.1 Constants

All computations use `mpmath` at `mp.dps = 60`. The three TOE primitives established in P209 are:

$$\mu = 4\pi^3 + \pi^2 + \pi \approx 137.036\,304, \quad (\text{monad, fine-structure inverse}) \quad (1)$$

$$e = \pi \approx 3.14159, \quad (\text{electron sector energy}) \quad (2)$$

$$\omega = \frac{\pi^3}{4} \approx 7.7523, \quad (\text{fundamental frequency } \Omega_0). \quad (3)$$

Their mutual relations are exact:

$$\omega = \frac{e^3}{4}, \quad \frac{\omega}{e^3} = \frac{1}{4} \quad (\text{exact}). \quad (4)$$

Two further constants enter via the  $G_2 \supset A_2$  structure (P211–P213):

$$E_\mu = \pi^2 = e^2, \quad h = h^\vee(A_2) = 3. \quad (5)$$

The geometric ratio and partial sum (P214, P215) are:

$$r = \frac{h E_\mu}{\mu^2} \approx 1.5767 \times 10^{-3}, \quad S_\infty = \frac{h \mu}{e^3(\mu^2 + h E_\mu)} \approx 7.049 \times 10^{-4}. \quad (6)$$

## 2.2 The closed-form formula

$$R_\infty = 207 \left( 1 - \frac{\omega}{e^3 \mu} + \frac{h \mu}{e^3 (\mu^2 + h E_\mu)} \right) \quad (7)$$

This is the closed sum of the geometric series  $R_\infty = 207(1 - T_0 + S_\infty)$  with  $T_k = (-r)^k \cdot (\mu/e^5)$  established in P215 Lemma 2.1, where  $T_0 = \omega/(e^3 \mu) = 1/4 \cdot e^0 \mu^{-1}$  is the zeroth correction.

## 2.3 Numerical verification at dps = 60

| Quantity                       | Value                           | Source      |
|--------------------------------|---------------------------------|-------------|
| $R_\infty$                     | 206.768 285 300 712 170 ...     | formula (7) |
| CODATA-2018                    | 206.768 283 0                   | [3]         |
| $\delta_\infty = R_\infty - T$ | $+2.3007 \times 10^{-6}$        | P216        |
| $\sigma_{\text{CODATA}}$       | $4.6 \times 10^{-6}$ (absolute) | [3]         |
| $\delta_\infty/\sigma$         | 0.500                           | P216        |
| Gap (ppm)                      | 0.0111 ppm                      | P216        |

The formula is zero-free-parameter: every symbol in (7) is either  $\pi$ , an integer, or an integer combination of powers of  $\pi$ . The 0.011 ppm gap is positive, structural, and within  $0.5\sigma$  of the CODATA-2018 measurement.

*Remark 2.1* (Interpretation of the positive gap).  $R_\infty > T_{\text{CODATA}}$  by  $2.30 \times 10^{-6}$ , i.e. the formula *overshoots* by half a measurement uncertainty. P216 established that  $\delta_\infty$  does not factorise cleanly in the natural TOE basis (PSLQ found no integer relation in a 10-element basis) and therefore represents a genuine structural constant of the  $G_2 \supset A_2$  embedding rather than a truncation artefact.

## 3 Axiom ID: Statement and Pedigree

### 3.1 Statement

**Axiom 3.1** (Root–Lepton Energy Identification). *Under the embedding  $G_2 \supset A_2$  used in the mass-ratio programme, the squared energy of the short roots of  $G_2$  is identified with the muon energy scale  $E_\mu = \pi^2$ , and the squared energy of the long roots of  $G_2$  is identified with the squared monad  $\mu^2 = (4\pi^3 + \pi^2 + \pi)^2$ . Equivalently, the per-root energy ratio is:*

$$\frac{|\alpha_{\text{short}}|^2}{|\alpha_{\text{long}}|^2} = \frac{E_\mu}{\mu^2} = \frac{r}{h} \approx 5.26 \times 10^{-4}. \quad (8)$$

Axiom ID is equivalent to identifying the  $A_2^{(s)}$  short-root subalgebra of  $G_2$  with the lepton face (muon–electron–tau sector) and the  $A_2^{(\ell)}$  long-root subalgebra with the monad sector. The ratio  $E_\mu/\mu^2 = r/h$  controls the convergence of the series  $S_\infty$ .

### 3.2 Where Axiom ID enters the derivation

Axiom ID enters at two points in the P186–P217 programme:

1. **The prefactor 207** (P211). The identification  $207 = \dim(G_2) \cdot (\dim(G_2) + 1) - h = 14 \cdot 15 - 3$  uses  $\dim(G_2) = 14$  and  $h = h^\vee(A_2) = 3$ . The  $A_2$  that appears here is the short-root subalgebra, and the identification of its dual Coxeter number with  $h = 3$  is a Lie-algebraic fact independent of Axiom ID.

2. **The exponent  $\pm 2$  pattern** (P217, Proposition P217.1). The  $G_2$  root-length ratio  $|\alpha_s|^2/|\alpha_\ell|^2 = 1/3 = 1/h$  is a Lie-algebraic fact. But the assignment  $|\alpha_s|^2 \leftrightarrow E_\mu$  and  $|\alpha_\ell|^2 \leftrightarrow \mu^2/h$  (which turns the abstract ratio into the physical ratio  $E_\mu/(\mu^2/h) = r$ ) is *precisely Axiom ID*.

Without Axiom ID, Proposition P217.1 establishes only that *some* energy scale assignment makes the  $\pm 2$  exponent pattern consistent with  $G_2 \supset A_2$  branching rules; it does not fix which physical scale sits on which root.

### 3.3 Is Axiom ID a consequence of R1–R8?

**The argument for:** R1–R8 fix the assembly of  $\hat{O}$ , and Paper 18 states that the spectrum of  $\hat{O}$  encodes fermion masses (§1.1 of Paper 18: “fermion masses from the discrete spectrum”). If the spectral resolution on  $S^3$  uniquely assigns lepton mass scales to sectors of the  $G_2$  root fan, then the short-root  $\leftrightarrow$  lepton identification follows from R1–R8 without additional input.

**The honest assessment:** Paper 18 is explicit (Remark 3.2 and Paper 18 Remark on mass-gap, §4.3) that the eigenvalue-based mass formula gives  $m_\mu/m_e \approx 3.1$  against the experimental  $\approx 206.8$  — a factor-66 discrepancy. The “slight degeneracy splitting” required to produce lepton mass ratios from the spectral approach *has not been computed in any paper in this corpus*. Paper 18 lists quantitative lepton mass ratios from pure geometry as “a major open problem.”

*Remark 3.2* (Paper 18 on its own mass formula). Paper 18 §4.3 states: “*The computed eigenvalue ratios give  $m_\mu/m_e \approx 3.1$ , against the experimental value of 206.8 — a factor-of-66 discrepancy. . . . Quantitative lepton mass ratios from pure geometry constitute a major open problem of this framework.*”

Given that Paper 18’s spectral rule does not explicitly produce the short-root  $\leftrightarrow$  lepton energy assignment, Axiom ID cannot currently be derived from R1–R8 within the published corpus. It is imported from the  $G_2 \supset A_2$  mass identification scheme developed across Addenda P186–P217, which is itself grounded in the TOE primitives  $(\mu, e, \omega)$  but not derived from the eigenvalue equation of  $\hat{O}$ .

**Conclusion:** Axiom ID is *consistent with* R1–R8 (it does not contradict any constraint) but is *not currently derived from* R1–R8. Theorem P218.1 is therefore stated as conditional on Axiom ID, with the proof obligation explicitly flagged as future work (Open Problem 6.1).

## 4 Programme Summary

The following table collects the seven principal results of the P18-T2 uniqueness programme (Addenda P186–P217).

| Label | Statement                                                                                                                                                                                            | Source | Status |
|-------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|--------|
| P1    | <b>Prefactor 207.</b> $207 = \dim(G_2) \cdot (\dim(G_2)+1) - h^\vee(A_2) = 14 \cdot 15 - 3$ . The prefactor is uniquely fixed by the $G_2 \supset A_2$ Lie-algebraic data.                           | P211   | PROVEN |
| P2    | <b>Correction numerator.</b> The first correction to 207 has numerator $\omega - h^\vee(A_2) = \pi^3/4 - 3$ , equal to the renormalised defect of the lepton face under $G_2 \supset A_2$ branching. | P210   | PROVEN |

| Label | Statement                                                                                                                                                                                                                                                                                                                                                                                                    | Source           | Status                        |
|-------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------|-------------------------------|
| P3    | <b>Uniqueness of <math>h^\vee = 3</math>.</b> The choice $h^\vee(A_2) = 3$ produces an agreement 208 times better than the next-best integer; no other integer substitution yields a sub-ppm result.                                                                                                                                                                                                         | P210             | PROVEN                        |
| P4    | <b><math>k</math>-independence of ratio <math>r</math>.</b> The ratio $T_{k+1}/T_k = -r$ with $r = hE_\mu/\mu^2$ is independent of $k$ ; exponent increments are exactly constant at each order.                                                                                                                                                                                                             | P214.1           | PROVEN                        |
| P5    | <b>Series factorisation.</b> $T_k = (-r)^k \cdot \sigma$ with $\sigma = \mu/e^5$ ; the series factorises into a seed $\sigma$ and a clean geometric ratio $(-r)^k$ .                                                                                                                                                                                                                                         | P215<br>Lem. 2.1 | PROVEN                        |
| P6    | <b>Exponent <math>\pm 2</math> from <math>G_2 \supset A_2</math>.</b> Each exponent increment $(\Delta n_e, \Delta n_\mu) = (+2, -2)$ is accounted for by the $G_2$ root-length ratio $ \alpha_s ^2/ \alpha_\ell ^2 = 1/h^\vee$ and the Coxeter factor, <i>given Axiom ID</i> . Without Axiom ID the argument establishes existence of a consistent assignment but not the specific physical identification. | P217.1           | CONJECTURED                   |
| P7    | <b>CODATA agreement.</b> $R_\infty = 206.768\,285\,300\dots$ lies within $\delta_\infty/\sigma = 0.500$ of the CODATA-2018 central value $m_\mu/m_e = 206.768\,283\,0$ , with gap 0.011 ppm.                                                                                                                                                                                                                 | P216             | (cond.<br>Ax. ID)<br>VERIFIED |

## 5 Theorem P218.1 (Main Theorem)

**Theorem 5.1** (P218.1 — P18-T2 Programme Closure). *Under Axiom 3.1 (Root–Lepton Energy Identification), the P18-T2 uniqueness programme is ESTABLISHED. More precisely:*

(i) *The closed-form formula*

$$R_\infty = 207 \left( 1 - \frac{\omega}{e^3 \mu} + \frac{h\mu}{e^3(\mu^2 + hE_\mu)} \right),$$

*with  $e = \pi$ ,  $\omega = \pi^3/4$ ,  $\mu = 4\pi^3 + \pi^2 + \pi$ ,  $h = 3$ ,  $E_\mu = \pi^2$ , is the unique zero-free-parameter formula consistent with  $G_2 \supset A_2$  structure at sub-ppm precision.*

(ii) *Every constant in  $R_\infty$  is a TOE primitive or integer: no free parameter appears.*

(iii)  $|R_\infty - T_{\text{CODATA}}|/\sigma_{\text{CODATA}} = 0.500 < 1$ . *The formula lies within one measurement uncertainty of CODATA-2018.*

(iv) *The seven results P1–P7 collectively account for every structural element of  $R_\infty$ : prefactor (P1), first correction numerator (P2), uniqueness of  $h$  (P3),  $k$ -independent ratio (P4), series factorisation (P5), exponent source (P6, conditional), and CODATA agreement (P7).*

- (v) *Axiom 3.1 is consistent with R1–R8 and is not contradicted by any constraint in Paper 18. Whether it follows from R1–R8 is an open proof obligation (Open Problem 6.1).*

*Proof sketch.* Part (i) follows from the uniqueness argument: the series  $207 \sum_{k=0}^{\infty} (-r)^k T_0$  is uniquely determined by P1–P5 given Axiom ID; its closed sum is  $R_{\infty}$  (P213, P215).

Part (ii) is immediate from the definitions:  $\pi$  is TOE-primitive (P209);  $h = 3$  is the dual Coxeter number of  $A_2$ , a Lie-algebraic integer;  $207 = 14 \cdot 15 - 3$  is fixed by  $\dim(G_2)$  and  $h$  (P211).

Part (iii) is a numerical computation at `mp.dps=60` confirmed by `verify_P218.py` (P7 / P216).

Part (iv) is the content of the programme summary in Section 4.

Part (v) follows from Section 3: the argument for and against is laid out there; the conditional declaration follows.  $\square$

**Corollary 5.2** (Uniqueness of assembly). *If Axiom 3.1 is established as a consequence of R1–R8, then Theorem 1.1 holds unconditionally: R1–R8 admit a unique admissible assembly of  $\hat{O}$ .*

## 6 Remaining Open Problems

Three open problems survive the closure of the programme.

**Open Problem 6.1** (OP-AxID: Derivation of Axiom ID from R1–R8). Show that the Root–Lepton Energy Identification (Axiom 3.1) follows from the coverage requirements R1–R8 of Paper 18, specifically from the spectral resolution of the master operator  $\hat{O}$  on  $S^3$ . This requires either (a) computing the degeneracy splitting that gives lepton mass ratios from the eigenvalue spectrum of  $\hat{O}$ , closing the factor-66 discrepancy noted in Paper 18 Remark 4.3; or (b) a direct representation-theoretic argument showing that the only  $G_2 \supset A_2$  sub-algebra assignment consistent with R1–R8 is the short-root  $\leftrightarrow$  lepton identification. Resolution would upgrade P6 from CONJECTURED to DERIVED and Theorem P218.1 from conditional to unconditional.

**Open Problem 6.2** (OP-rSinf-100: Near-miss  $r \cdot S_{\infty} \approx c/100$ ). Let  $c = \delta_{\infty}/207$  and  $\varepsilon = c/(r \cdot S_{\infty}) - 1/100$ . Numerically,

$$\varepsilon \approx -2.204 \times 10^{-7}.$$

PSLQ at `mp.dps=60` finds no integer relation for  $\varepsilon$  in a 7-element natural basis. The near-miss  $r \cdot S_{\infty} \approx c/100$  is therefore either a deep identity at higher order or a numerical coincidence. Resolve which, or find the exact algebraic form of  $\varepsilon$ .

**Open Problem 6.3** (OP-G2A2-lattice: Adjacency resolvent mismatch  $\times 97.7$ ). The unweighted  $A_2$  adjacency resolvent evaluated at the TOE energy scale  $\lambda = \mu/e$  gives a value  $\approx 97.7$  times larger than  $S_{\infty}$  (P215, Section 5). A weight-lattice or holonomy integral over the  $A_2$  face that produces  $S_{\infty}$  exactly from the adjacency data of the  $G_2/A_2$  coset has not been found. Identifying this mechanism would close OP-G2A2-lattice and potentially provide an independent derivation of  $S_{\infty}$  from Lie-algebraic data alone.

**Open Problem 6.4** (OP-Exactness: Future measurement).  $R_{\infty}$  overestimates CODATA-2018 by  $2.30 \times 10^{-6}$  (0.011 ppm). The CODATA-2018 uncertainty is  $\sigma = 4.6 \times 10^{-6}$ , so the discrepancy is  $0.5\sigma$ . A future measurement of  $m_{\mu}/m_e$  with  $\sigma < 10^{-6}$  would either confirm that  $R_{\infty}$  is the exact theoretical value (if the measurement moves toward  $R_{\infty}$ ) or falsify the formula (if the gap exceeds  $3\sigma$ ). The AMM and LEPTON-G programmes are the most likely candidates for such precision.

## References

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