

$\text{sym}E_e$

Addendum 214: OP-G₂A₂-ratio Deriving the Geometric Ratio $r = h^\vee(A_2) E_\mu \alpha^2$

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Context. Addendum 213 (P213) established that the G_2 - A_2 muon mass-ratio correction series $\{T_k\}_{k \geq 1}$ is a geometric series with common ratio

$$= \left. \frac{T_{k+1}}{T_k} \right|_{\text{sign-stripped}} = \frac{h^\vee(A_2) E_\mu}{\mu^2} = \frac{3\pi^2}{\mu^2} \approx 1.577 \times 10^{-3}, \quad (1)$$

verified to 10^{-63} precision at `mp.dps = 60`. The individual correction terms are

$$T_k = (-1)^{k+1} h^k$$

, $k \geq 1$, with 0 from G_2 - A_2 geometry and/or loop-correction structure.

The present addendum addresses this question in four sections:

1. numerics and TOE interpretation of r ;
2. algebraic derivation of r from the T_k formula (Proposition P214.1 — proven);
3. three geometric derivation attempts (Conjecture P214.2 — conjectured);
4. closed-form structure of R_∞ and the deformed-monad denominator.

1 The Ratio in TOE Terms

1.1 Numerical value and equivalent forms

Setting $= 1/\mu$ (the fine-structure constant), the ratio admits several equivalent expressions:

$$= \frac{3\pi^2}{\mu^2} = h^\vee(A_2) \cdot E_\mu \cdot^2 = h^\vee(A_2) \cdot E_e^2 \cdot^2, \quad (2)$$

all verified to full `mp.dps = 60` precision (see `verify_P214.py`, assertions A1–A10):

$$= 0.001\,576\,702\,397\,389\,542\,007\,6\dots$$

1.2 TOE interpretation

The formula $r = h \cdot E_e^2 \cdot^2$ decomposes into three factors of clear TOE provenance:

- $h = h^\vee(A_2) = 3$ is the dual Coxeter number of A_2 , the Lie algebra governing the lepton face of the G_2 spectral frame.
- $E_e^2 = \pi^2 = E_\mu$ is the muon-sector energy scale. In the TOE identification rule, the electron sector carries energy $E_e = \pi$ at each site. The muon sector energy E_μ is the squared electron-sector energy: $E_\mu = E_e^2$. Equivalently, E_μ is the energy accumulated by traversing *two* electron-sector edges in sequence.
- $\cdot^2 = 1/\mu^2$ is the square of the fine-structure constant. Each loop correction in the G_2 – A_2 spectral expansion carries one power of $\cdot^2$, the standard two-vertex suppression of quantum corrections.

Thus r is the product of an algebraic A_2 -face count, a two-step energy weight, and a two-loop coupling suppression. It is not a free parameter: every factor is fixed by the TOE geometry.

2 Algebraic Derivation from the T_k Formula

Proposition 2.1 (P214.1 — Geometric ratio, algebraic proof). *Let*

$$T_k = (-1)^{k+1} h^k \text{ independent of } k.$$

Then the three exponent increments are :

h -exponent: $(k+1) - k = +1$ in numerator, giving factor h .

0 exponent: all increments are constant, the ratio $T_{k+1}/T_k = -hE_\mu/\mu^2$ is k -independent, establishing the geometric series structure.

Remark 2.2. Proposition P214.1 is a *formal* result: it follows purely from the exponent pattern in the T_k formula and does not yet explain *why* the exponents of 0 *are* addressed in Section 3.

Corollary 2.3. *The full geometric series $\sum_{k=1}^{\infty} T_k = T_1 \sum_{j=0}^{\infty} (-)^j$ converges absolutely since $\approx 1.577 \times 10^{-3} \ll 1$, and its sum is*

$$S_{\infty} = \frac{T_1}{1+} = \frac{h\mu}{1+}$$

3 Geometric Derivation Attempts

We now ask why the specific increment pattern (0es.Threeindependentlinesofattackareexplored).

3.1 Attempt 1: A root-system lengths

The A_2 root system has two root lengths (both equal in the standard normalization). In the G_2 Killing form with long roots at $\text{length}^2 = 2$, the A_2 subalgebra consists of the long roots of G_2 ($\text{length}^2 = 2$), while the short roots have $\text{length}^2 = 2/3$.

A natural candidate for r via the Killing form is:

$$Killing = h^\vee(A_2) \cdot |\alpha_{long}|_{G_2}^2 \cdot 2 = 3 \cdot 1 \cdot 2 \approx 1.598 \times 10^{-4}.$$

Assessment: *This is a factor of $E_\mu = \pi^2 \approx 9.87$ too small. The Killing-form normalisation of root lengths does not explain the factor E_μ . The loop-suppression per A_2 -face edge is , not $\cdot |\alpha_{root}|$ in this normalisation.*

3.2 Attempt 2: Monad-energy scaling

In the TOE, $\mu = 4\pi^3 + \pi^2 + \pi$ is the monad: the unique rational polynomial in π whose reciprocal equals the fine-structure constant. Its square μ^2 naturally suppresses two-loop amplitudes. The question reduces to: why does each successive T_k carry an extra factor of $E_\mu/\mu^2 = \pi^2/\mu^2$ relative to T_{k-1} ?

In TOE spectral theory, the G_2 frame assigns each site one unit of E_e . A two-site amplitude (traversal of one face edge plus return) carries $E_e^2 = E_\mu$. Suppressed by one power of $2 = 1/\mu^2$ (the two-vertex coupling), such an amplitude contributes $E_\mu \cdot 2 = E_\mu/\mu^2$. Multiplied by $h = h^\vee(A_2) = 3$, this is exactly .

Assessment: *This argument is internally consistent and makes quantitative contact with r , but it assumes the amplitude assignment E_e per site and 2 per two-vertex insertion. These assignments are the TOE identification rule (not derived here from the G_2 geometry itself). This attempt is therefore not yet a derivation from first principles, but rather a restatement of r in spectral-theory language.*

3.3 Attempt 3: Two-step paths on the A lepton triangle

The A_2 root system realised as a lepton face of G_2 has exactly three positive roots, forming a triangle with three vertices and three edges. In the TOE, each edge carries energy weight $E_e = \pi$. A two-step path traverses two consecutive edges, accumulating energy $E_e^2 = E_\mu$.

Path count. *On the A_2 triangle with vertices $\{v_1, v_2, v_3\}$:*

- *Oriented directed 2-step paths ($v_i \rightarrow v_j \rightarrow v_k$, $j \neq i$, $k \neq j$): there are $3! = 6 = |W(A_2)|$ such paths.*
- *Unoriented 2-step paths (identifying each path with its reversal): there are $3 = h^\vee(A_2)$ such paths.*

Each unoriented 2-step path contributes energy weight E_μ and coupling suppression 2 per pair of vertices. Summing over all three unoriented 2-step paths gives:

$$= h^\vee(A_2) \cdot E_\mu \cdot 2 = 3\pi^2 \approx 1.577 \times 10^{-3}.$$

This matches the ratio exactly (verified to 10^{-60} in `verify_P214.py`, assertions A21–A25).

Assessment: *The path count $h^\vee(A_2) = 3$ equals the number of unoriented 2-step paths on the A_2 triangle, and the energy weight E_μ equals E_e^2 (the energy of one 2-step traversal).*

This gives a clean combinatorial derivation: each order in the loop expansion corresponds to one traversal of the lepton triangle by 2-step paths, with the number of independent traversals set by the dual Coxeter number. The coupling suppression ² per traversal is the standard two-vertex insertion rule. This attempt achieves the strongest geometric contact with r .

Conjecture 3.1 (P214.2 — Two-step path origin of r). *Each order k of the G_2 – A_2 mass-ratio loop expansion corresponds to $h^\vee(A_2)$ unoriented 2-step path traversals of the lepton face triangle, each carrying energy weight $E_e^2 = E_\mu$ and coupling suppression ². The ratio of consecutive correction terms is therefore:*

$$\frac{T_{k+1}}{T_k} = -h^\vee(A_2) \cdot E_\mu \cdot^2 = -.$$

The specific proof obligation is: derive the identification of “one loop order” with “one set of $h^\vee(A_2)$ 2-step traversals” directly from the G_2 – A_2 branching rules and/or the spectral resolution of the Dirac operator on the lepton face.

4 Closed-Form Structure

4.1 The infinite sum in factored form

The complete mass-ratio formula is:

$$R_\infty = 207 \left(1 - \frac{\omega}{\mu^2 + hE_\mu} \right), \quad (3) \text{ where } 207 = 14 \cdot 15 - h^\vee(A_2) = \dim(G_2) \cdot (\dim(G_2) + 1) - 3 \text{ is the prefactor from the } G_2 \text{ spectral frame. Numerically:}$$

$$R_\infty = 206.768\,285\,300\,712\,17\dots, \quad |R_\infty - T|/T \approx 0.01113 \text{ ppm}.$$

4.2 The deformed-monad denominator

The denominator $\mu^2 + hE_\mu$ is the monad squared deformed by the A_2 face contribution:

$$\mu^2 + hE_\mu = \mu^2(1+) = \mu^2 + 3\pi^2. \quad (4)$$

This identity follows directly from $= hE_\mu/\mu^2$ (see `verify_P214.py`, assertion A26).

4.3 Numerator expansion

Writing the bracket of (4.1) over the common denominator $0ing0o$ that $0equals79\,827\,829.477\,966\dots$, verified

Remark 4.1. The factor $\omega(4\mu - 1)$ in (??) reflects the TOE structure *0implification* $\omega(4\mu - 1) = 3\omega(\dots)$ would require $4\mu - 1 = 3$, i.e. $\mu = 1$, which is false ($\mu \approx 137.04$). The expression (??) is the correct minimal form.

4.4 Alternative factoring via $(1 + r)$

$$\text{Using } \mu^2 + hE_\mu = \mu^2(1+) \text{ and } T_1 = h/(0oR_\infty = 207 \left(1 + T_0 + \frac{T_1}{1+} \right), \quad T_0 = -\frac{\omega}{\mu^2} \quad (5)$$

This makes explicit that the closed form is the leading-term approximation $207(1 + T_0 + T_1)$ corrected by the geometric resummation factor $1/(1+)$.

5 Status of OP-G2A2-ratio

Proven (Proposition P214.1): The ratio $r = T_{k+1}/T_k = -hE_\mu/\mu^2$ follows algebraically from the exponent structure of the T_k formula. The three exponent increments (+1 in h , +2 in 0 tantacross all k , establishing the unique k -independent geometric ratio. This is a complete, rigorous proof given the formula.

Conjectured (Conjecture P214.2): The geometric origin of the exponent- ± 2 shift in 0 remains open. Attempts to show that each loop order corresponds to $h^\vee(A_2) = 3$ unoriented 2-step traversals of the lepton face triangle (energy weight $E_e^2 = E_\mu$, coupling suppression 2). This model exactly reproduces r but has not been derived from the G_2 - A_2 spectral decomposition.

Status: CONJECTURED.

Specific proof obligation: Derive the formula $T_k = (-1)^{k+1}h^k 0$ from principles either from the G_2 - A_2 branching rules, from the spectral resolution of a Dirac-type operator on the lepton face, or from a loop-integral over the A_2 root system — without assuming the pattern. Equivalently, show that each perturbative order on the A_2 lepton face contributes exactly $(-hE_\mu/\mu^2)$ times the previous order, using only the Lie-algebraic data of $G_2 \supset A_2$ and the TOE identification rule $E_e = \pi$, $\mu = 4\pi^3 + \pi^2 + \pi$.