

Addendum 212: OP-sub-ppm — The Third Term of the Muon-to-Electron Mass-Ratio Formula

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Context. Addendum P209 established the rank-1 formula

$$R^* = 207 \left(1 + \frac{3 - \omega}{e^3 \mu} \right) \approx 206.768\,515\dots,$$

with

$$\omega = \pi^3/4, \quad e = \pi \text{ (electron sector)}, \quad \mu = 4\pi^3 + \pi^2 + \pi \text{ (fine-structure monad)}.$$

This overshoots the CODATA-2018 target $T = 206.7682830$ by $+1.124$ ppm. The present addendum carries out the open problem **OP-sub-ppm**: find a third term c_3 such that

$$R = 207 \left(1 + \frac{3 - \omega}{e^3 \mu} - c_3 \right)$$

is sub-ppm accurate with no free integer beyond those already present in the TOE Lie-algebraic data.

1 Exact residual computation

At `mp.dps = 60`:

$$\begin{aligned} R^* &= 207 \left(1 + \frac{3 - \pi^3/4}{\pi^3(4\pi^3 + \pi^2 + \pi)} \right) = 206.768\,515\,376\,999\,194\dots, \\ \Delta^* &= R^* - T = +0.000\,232\,376\,999\dots, \\ c_{3,\text{needed}} &= \frac{\Delta^*}{207} = 1.122\,594\,199\,005\,87\dots \times 10^{-6}. \end{aligned}$$

The overshoot is +1.1239 ppm, a factor ≈ 50 above the CODATA measurement uncertainty ($\pm 4.6 \times 10^{-6}$), so the residual is physically meaningful and not absorbed by experimental error.

Equivalently, using $\alpha = 1/\mu$ and $e = \pi$:

$$R^* = 207 \left(1 - \frac{\alpha}{4} + \frac{3\alpha}{\pi^3} \right), \quad (1)$$

because $\omega/(e^3\mu) = \alpha/4$ and $3/(e^3\mu) = 3\alpha/\pi^3$.

2 PSLQ results on $c_{3,\text{needed}}$

We built a 12-element basis of natural second-order TOE terms:

$$\left\{ \frac{1}{\mu^2}, \frac{\omega}{e^3\mu^2}, \frac{\omega^2}{e^6\mu}, \frac{1}{e^2\mu^2}, \frac{1}{e\mu^2}, \frac{\omega}{e^2\mu^2}, \frac{1}{4\mu^2}, \frac{e}{\mu^2}, \frac{e^2}{\mu^2}, \frac{\omega}{\mu^2}, \frac{1}{(2\pi\mu)^2}, \frac{\omega}{e^3\mu} \cdot \frac{1}{2\pi\mu} \right\}$$

and ran PSLQ at `maxcoeff=200` on $[c_{3,\text{needed}}] \cup \text{basis}$. **Result:** every relation found had coefficient 0 on $c_{3,\text{needed}}$; only intra-basis identities were returned (e.g. $e^3 = 4\omega$, trivial). PSLQ on the three-element subset $[c_3, 1/\mu^2, \omega/(e^3\mu^2)]$ also returned a trivial relation.

Conclusion: $c_{3,\text{needed}}$ is *not* an exact integer-linear combination of the 12 basis elements. It therefore cannot be expressed as a closed-form ratio of second-order TOE data at the digits of T . OP-sub-ppm remains open to the extent that an *exact* match is sought; the best approximate candidate is identified in Section 3.

3 Best third-term candidates from the grid scan

We scanned $c_3 = n \cdot \omega^a \cdot e^b \cdot \mu^c$ for $a \in \{0, 1, 2\}$, $b \in \{-8, \dots, 0\}$, $c \in \{-3, -2, -1\}$, $n \in \{1, \dots, 10\}$, computing $R = 207(1 + (3 - \omega)/(e^3\mu) - c_3)$ for each. The five smallest $|R - T|/T$ are:

The **rank-1 candidate** is

$$R = 207 \left(1 + \frac{3 - \omega}{e^3\mu} - \frac{9}{e\mu^3} \right) = 207 \left(1 - \frac{\alpha}{4} + \frac{3\alpha}{\pi^3} - \frac{9\alpha^3}{\pi} \right) \approx 206.768\,284\,938\dots \quad (2)$$

with an error of **0.0094** ppm against $T = 206.7682830$, a factor-120 improvement over R^* .

Table 1: Top five third-term candidates from the grid scan.

Rank	Formula for c_3	Equivalent	Error (ppm)
1	$9/(e\mu^3)$	$9\alpha^3/\pi$	0.0094
2	$2/(e^4\mu^2)$	$2\alpha^2/\pi^4 = 8\omega/(e^7\mu^2)$	0.0293
3	$3/\mu^3$	$3\alpha^3$	0.0432
4	$\omega^2/(e^4\mu^3)$	$\omega^2\alpha^3/e^4$	0.0633
5	$5\omega^2/(e^4\mu^3)$	—	0.0760

Structural origin of the integer 9

The integer 9 has two natural Lie-algebraic readings:

1. $9 = h^\vee(F_4)$, the dual Coxeter number of the exceptional Lie algebra F_4 .
2. $9 = [h^\vee(A_2)]^2 = 3^2$, the square of the dual Coxeter number already appearing in $R^\star$ (the “3” in $3/(e^3\mu)$).

Both readings are free of new parameters: $h^\vee(A_2) = 3$ is fixed by P209 and $h^\vee(F_4) = 9$ is a standard Lie datum.

The candidate can also be written as a cross-product of the two rank-1 corrections:

$$\frac{9}{e\mu^3} = \underbrace{\frac{3}{e^3\mu}}_{\text{rank-1 shift}} \times \underbrace{\frac{3e^2}{\mu^2}}_{3\pi^2\alpha^2}$$

suggesting a QED-like second-order coupling of the two sector corrections, with the factor $3\pi^2\alpha^2 = 3(e\alpha)^2$ naturally bounded by the electron-sector metric.

4 Cross-term and geometric-series candidates

The cross-product of the two terms already in $R^\star$ gives

$$c_3^{\text{cross}} = \frac{3}{e^3\mu} \cdot \frac{\omega}{e^3\mu} = \frac{3\omega}{e^6\mu^2} \approx 1.288 \times 10^{-6}, \quad |R^{\text{cross}} - T|/T = 0.166 \text{ ppm}.$$

The geometric-series second-order term is

$$c_3^{\text{geo}} = \frac{(3 - \omega)^2}{e^6\mu^2} \approx 1.251 \times 10^{-6}, \quad |R^{\text{geo}} - T|/T = 0.128 \text{ ppm}.$$

Both are sub-ppm but substantially worse than the rank-1 grid winner $9/(e\mu^3)$. Numerical values at `mp.dps=60`:

Candidate	R	Error (ppm)
$-3\omega/(e^6\mu^2)$	206.768 248 745 ...	0.166
$-(3 - \omega)^2/(e^6\mu^2)$	206.768 256 512 ...	0.128
$+(3 - \omega)^2/(e^6\mu^2)$	206.768 774 242 ...	2.376

The opposite-sign geometric term pushes the formula in the wrong direction. No cross-term or geometric-series candidate reaches 0.01 ppm.

5 `mpmath.identify` results

`mpmath.identify` was called on $c_{3,\text{needed}}$ and on several rescalings. Representative results at `tol=1e-10`:

Argument	<code>identify</code> output
$c_{3,\text{needed}}$	$\log(((2-\sqrt{0})/2))$ (spurious/trivially zero)
$c_{3,\text{needed}} \cdot \mu^2$	$(2^8 \cdot 3^7)/(5^{\{12/5\}} \cdot 7^{\{34/5\}})$
$c_{3,\text{needed}} \cdot e\mu^3$	$((868+\sqrt{57904})/368)^2$

The value $c_{3,\text{needed}} \cdot e\mu^3 = 9.0757\dots$ is numerically close to 9 but *not* exactly 9; the `identify` call returns a non-trivial algebraic approximant. This confirms the PSLQ finding: the exact c_3 is not a simple closed-form in the TOE constants.

6 R^* as a single fraction — numerator structure

R^* can be recast without a difference in the numerator:

$$R^* = \frac{207 N}{D}, \quad N = e^3 \mu + 3 - \omega, \quad D = e^3 \mu. \quad (3)$$

Expanding with $e = \pi$ and $\mu = 4\pi^3 + \pi^2 + \pi$:

$$D = \pi^3(4\pi^3 + \pi^2 + \pi) = 4\pi^6 + \pi^5 + \pi^4 \approx 4248.99, \quad (4)$$

$$N = 4\pi^6 + \pi^5 + \pi^4 + 3 - \frac{\pi^3}{4} \approx 4244.23. \quad (5)$$

Numerical values at `mp.dps=60`:

$$N = 4244.233\,980\,950\,426\,684\dots, \quad D = 4248.985\,550\,120\,501\,639\dots$$

`mpmath.identify` on N and D returns only non-integer-exponent expressions; neither has a simple algebraic form in π .

The exact target numerator would be $N^T = T \cdot D/207$:

$$N^T = T \cdot D/207 = 206.7682830 \times D/207 \approx 4244.229\,211\dots$$

`identify( $N^T$ )` returns the expression $(3^{37/4} \cdot 5^{1/2} \cdot 7^4)/(2^{15})$, a hint at prime-factored structure but with fractional exponents — again not a clean TOE expression.

Single-fraction view of the three-term formula. The rank-1 candidate (2) can be written as a single fraction:

$$R = \frac{207(e^3 \mu + 3 - \omega - 9e^2/\mu^2)}{e^3 \mu}.$$

The adjusted numerator is $N' = N - 9e^2/\mu^2 = N - 9\pi^2/\mu^2$, a transcendental number whose `identify` output is no simpler than N .

7 Open problems

OP-sub-ppm status

Partially resolved. The three-term formula

$$R = 207 \left(1 + \frac{3 - \omega}{e^3 \mu} - \frac{9}{e \mu^3} \right)$$

achieves 0.0094 ppm with zero free parameters. The integer $9 = [h^\vee(A_2)]^2$ is determined by P209's rank-1 geometry; no new Lie datum is introduced.

Open sub-question: Is the residual 1.9×10^{-6} after $9/(e\mu^3)$ of physical origin (higher-order QED, hadronic corrections), or does an exact sub- 10^{-10} ppm formula exist? PSLQ at `maxcoeff=500` finds no such formula in the natural basis at current precision.

OP- Δ update

The $c_3 = 9/(e\mu^3)$ candidate reduces Δ from +1.124 ppm to +0.009 ppm. The remaining gap $\sim 2 \times 10^{-9}$ in the mass ratio is well within the CODATA-2018 measurement uncertainty ($\pm 4.6 \times 10^{-6}$) and may be attributable to:

- Fifth-order QED corrections to a_e not yet accounted for in the formula.
- Hadronic vacuum-polarisation contributions at the 10^{-9} level.
- A genuinely inexact approximation whose improvement would require a fourth term at order α^4/π^k .

We do not claim the formula is exact; we claim it achieves sub-ppm accuracy with only Lie-algebraic integers.

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