

Addendum P210

OP-3 Resolution Attempt: The Dual Coxeter Number $h^\vee(A_2) = 3$ and the Lepton Face Mass-Ratio Correction Why the Muon/Electron Formula Selects the Triangular Topology of the A_2 Root System

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1 Setup and Restatement

Status: OPEN. The integer 3 in $R^\star$ is identified as $h^\vee(A_2) = 3$, the dual Coxeter number of the root system carried by the lepton face of the 4-simplex on S^3 . No proof of this identification has been established; the structural argument of Section 4 reaches conjecture level only.

1.1 Background from prior addenda

The prior addenda contributing to this programme are:

- **P202:** The 4-simplex inscribed in S^3 has inner product $\langle v_i, v_j \rangle = -\frac{1}{4}$ for all pairs $i \neq j$, geodesic angle $\theta_4 = \arccos(-\frac{1}{4})$, and f -vector $(5, 10, 10, 5)$.
- **P203:** Of the 10 triangular 2-faces, all carry A_2 -type geometry. The dual Coxeter number $h^\vee(A_2) = 3$ appears as the coupling ratio in the mass formula; equidistance between vertices ($\langle v_i, v_j \rangle = \text{const}$) is the geometric realisation of the No-Harkonnen pin and ensures heat conservation across Fork/Weld cycles.
- **P208:** The lepton face is the triangular 2-face $\{\text{Plateau} \leftrightarrow v_2, \text{Weld} \leftrightarrow v_1, \text{Fork} \leftrightarrow v_0\}$, a triangular 2-face of A_2 type. The Fork-dominant energy $E_F = 4\pi^3 \gg E_W = \pi^2$.
- **P209:** The formula

$$R^\star = 207 \left(1 + \frac{3 - \omega}{e^3 \mu} \right) = 207 \left(1 - \frac{\omega - 3}{e^3 \mu} \right) \approx 206.768\,515 \dots \quad (1)$$

achieves 1.124 ppm agreement with CODATA (206.768 283), using only the three TOE primitives $\mu, e = \pi, \omega = \pi^3/4$ and the integer 3.

1.2 TOE primitives

All computations use `mpmath` at `mp.dps = 60`.

$$\mu = 4\pi^3 + \pi^2 + \pi \approx 137.036\,304 \quad (\text{monad / fine-structure inverse}), \quad (2)$$

$$e = \pi \approx 3.141\,593 \quad (\text{electron sector energy / } E_e), \quad (3)$$

$$\omega = \frac{\pi^3}{4} \approx 7.751\,569 \quad (\text{fundamental frequency, } \Omega_0). \quad (4)$$

Their mutual relation $\mu = 4e^3 + e^2 + e$ makes μ a polynomial in e with integer coefficients. The relation $\omega = e^3/4$ is exact, giving $e^3 = 4\omega$ and $e^3\mu = 4\omega\mu$.

1.3 Restatement in structural terms

The question addressed by this addendum is:

Can the integer 3 in $R^\star$ be identified as $h^\vee(A_2) = 3$, where $h^\vee$ is the dual Coxeter number of the A_2 root system carried by each triangular face of the 4-simplex?

With this identification the formula reads

$$R^\star = 207 \left(1 - \frac{\omega - h^\vee(A_2)}{e^3 \mu} \right) \quad (5)$$

and the correction $(\omega - h^\vee)/(e^3\mu)$ is the ratio of *excess fundamental frequency over dual Coxeter number* to *cubic electron energy times monad*.

2 Geometric Analysis of $(\omega - h^\vee(A_2))$

2.1 Exact value

Using $\omega = \pi^3/4$ and $h^\vee(A_2) = 3$:

$$\omega - h^\vee(A_2) = \frac{\pi^3}{4} - 3 \approx 4.751\,569\,170\,074\,955. \quad (6)$$

Multiplying by 4:

$$4(\omega - h^\vee) = \pi^3 - 12 = e^3 - 4h^\vee(A_2) \approx 19.006\,277. \quad (7)$$

This exposes the correction as proportional to $e^3 - 4h^\vee$, measuring how far the cube of the electron energy exceeds four times the dual Coxeter number.

2.2 Key reformulation

Using $e^3 = 4\omega$ the denominator simplifies:

$$\frac{\omega - h^\vee}{e^3 \mu} = \frac{\omega - h^\vee}{4\omega \mu} = \frac{1}{4\mu} \left(1 - \frac{h^\vee}{\omega} \right) = \frac{1}{4\mu} \left(1 - \frac{12}{\pi^3} \right). \quad (8)$$

Numerically:

$$\frac{h^\vee(A_2)}{\omega} = \frac{3}{\pi^3/4} = \frac{12}{\pi^3} \approx 0.387\,018. \quad (9)$$

This ratio is not a standard constant of physics; it is a purely geometric quantity measuring what fraction of the fundamental frequency is “used up” by the dual Coxeter number of the triangular face.

2.3 Defect $1 - h^\vee/\omega$

The quantity

$$\delta \equiv 1 - \frac{h^\vee(A_2)}{\omega} = 1 - \frac{12}{\pi^3} \approx 0.612\,982 \quad (10)$$

is the fractional excess of ω over $h^\vee$. The correction becomes $\delta/(4\mu)$. No identification of δ with a Lie-algebraic or spectral quantity has been found; it is recorded here as an open datum.

2.4 Euler characteristic of the lepton face

The lepton triangle (boundary only) has $V = 3$ vertices and $E = 3$ edges, giving Euler characteristic

$$\chi_\partial = V - E = 3 - 3 = 0. \quad (11)$$

Including the interior face: $\chi = V - E + F = 3 - 3 + 1 = 1$ (contractible disk). The boundary Euler characteristic $\chi_\partial = 0$ is consistent with the A_2 root system having $h^\vee = 3 =$ the number of vertices $=$ the number of edges, making the triangle a balanced object in which no vertex or edge class is privileged.

2.5 Gram matrix and eigenvalue structure

The 3×3 Gram matrix G of the lepton face, with $G_{ii} = 1$ and $G_{ij} = \langle v_i, v_j \rangle = -\frac{1}{4}$ for $i \neq j$, is a circulant. Its eigenvalues are computed analytically:

$$\lambda_{\max} = 1 - (-\frac{1}{4}) = \frac{5}{4} \quad (\text{multiplicity } 2), \quad \lambda_{\min} = 1 + 2(-\frac{1}{4}) = \frac{1}{2} \quad (\text{multiplicity } 1). \quad (12)$$

Trace check: $2 \cdot \frac{5}{4} + \frac{1}{2} = 3 = V$. Determinant: $(\frac{5}{4})^2 \cdot \frac{1}{2} = \frac{25}{32}$.

The minimal eigenvalue $\lambda_{\min} = \frac{1}{2}$ belongs to the uniform eigenvector $\frac{1}{\sqrt{3}}(1, 1, 1)$; the two raised eigenvectors span the orthogonal complement. The trace equals the vertex count $V = h^\vee(A_2) = 3$.

2.6 Heat budget on the lepton face

The vertex energies from P208 are $E_F = 4\pi^3$, $E_W = \pi^2$, $E_P = \pi$, with total $E_F + E_W + E_P = \mu$. The heat budget is:

$$E_{\text{Fork}} : E_{\text{Weld}} : E_{\text{Plateau}} = 4\pi^3 : \pi^2 : \pi = 4\pi^2 : \pi : 1. \quad (13)$$

Fork dominates ($E_F/\mu \approx 90.5\%$), consistent with $E_F = 4\pi^3 = e^3/4 \cdot 16 = 16\omega$ greatly exceeding the other two. The three vertex energies are not equal; the equidistance is in the inner product geometry (all $\langle v_i, v_j \rangle = -1/4$), not in the energy assignment. Heat conservation is guaranteed by the Wheel's Fork $\leftrightarrow$ Weld cycle operating on this face.

3 Integer Scan: $n = 1, \dots, 10$

The family of candidate formulas indexed by integer n is:

$$R(n) = 207 \left(1 + \frac{n - \omega}{e^3 \mu} \right). \quad (14)$$

We compare each to the CODATA value 206.768 283 0 in parts per million.

n	$R(n)$	$ R(n) - \text{TARGET} /\text{TARGET}$ [ppm]	Lie-algebraic meaning
1	206.671 080	470.104	—
2	206.719 798	234.490	$\text{rank}(A_2)$
3	206.768 515	1.124	$h^\vee(A_2) = h(A_2) = \Phi^+(A_2) $
4	206.817 232	236.738	—
5	206.865 950	472.352	—
6	206.914 668	707.966	$ W(A_2) = \Phi(A_2) $
7	206.963 385	943.580	—
8	207.012 103	1179.194	$\dim(A_2)$
9	207.060 820	1414.808	—
10	207.109 538	1650.422	—

Table 1: Gap table for $R(n) = 207(1 + (n - \omega)/(e^3\mu))$ versus CODATA. The unique minimum occurs at $n = 3 = h^\vee(A_2)$.

The minimum is sharp and isolated at $n = 3$. The second-closest candidate, $n = 2$ ($\text{rank}(A_2)$), is 208 times worse. Among all Lie-algebraic invariants of A_2 — rank, Coxeter number, dual Coxeter number, number of positive roots, Weyl group order, total root count, dimension — only $h^\vee(A_2) = h(A_2) = |\Phi^+(A_2)| = 3$ selects the sub-ppm regime. (For A_2 , all three of these invariants coincide at 3; no test can distinguish them on this basis alone.)

Remark on the A_2 triple coincidence. For $A_2 \cong \mathfrak{su}(3)$:

$$h^\vee(A_2) = h(A_2) = |\Phi^+(A_2)| = 3. \quad (15)$$

This triple coincidence holds only for A_n (where $h = h^\vee = n + 1$ and $|\Phi^+| = n(n + 1)/2$; equal to $h^\vee$ only for $n = 2$). The scan cannot resolve which of the three is “really” the relevant invariant. P203 gives independent evidence for $h^\vee$ via the heat-conservation argument; we adopt $h^\vee(A_2)$ as the primary label throughout.

4 Structural Argument

We state the best available structural argument for the identification $3 = h^\vee(A_2)$ in R^* . We label this a conjecture; the proof obligations are stated explicitly.

4.1 Context

The mass ratio formula arises from the spectral weight of the lepton sector in the 4-simplex. The leading term 207 selects the lepton face (the Fork/Weld/Plateau triangle, three of the five simplex vertices). The correction term is a ratio of frequencies and energies derived from the same geometric objects.

Conjecture 4.1 (OP-3: Dual Coxeter coupling). The integer 3 appearing in the mass ratio correction $(\omega - 3)/(e^3\mu)$ is the dual Coxeter number $h^\vee(A_2) = 3$ of the root system carried by each triangular 2-face of the 4-simplex on S^3 .

4.2 Supporting evidence

(i) Integer scan uniqueness. Of all integers $n = 1, \dots, 10$, only $n = 3$ gives sub-10-ppm agreement with CODATA; the gap of 1.124 ppm at $n = 3$ is $200\times$ better than the next integer.

(ii) **A_2 face type.** P202–P203 established that every triangular 2-face of the 4-simplex carries the A_2 root system. The correction involves the lepton face directly; it is structurally consistent that the Lie-algebraic invariant of that face appears in the formula.

(iii) **Dual Coxeter number as coupling.** In gauge theory, one-loop corrections to propagators carry factors of $h^\vee(G)$ for the gauge group G . For $G = SU(3)$, i.e. A_2 , $h^\vee = 3$. The correction $(\omega - h^\vee)/(e^3\mu)$ is formally analogous to a one-loop correction with coupling ω normalised by $h^\vee$: the face carries a “loop weight” $h^\vee$ which is subtracted from the bare frequency ω before the correction is divided out by the cubic energy-scale $e^3\mu$.

(iv) **Heat conservation.** P203 showed that equidistance ($\langle v_i, v_j \rangle = -\frac{1}{4}$ for all pairs) is the geometric form of the No-Harkonnen pin. The A_2 Gram matrix (Section 2) with all three pairs equal realises this exactly. The dual Coxeter number $h^\vee = 3$ counts the number of positive roots, which equals the number of face edges (3), which equals the number of vertices (3): the triangle is “balanced” in the sense that no pairing is privileged. This balance is what heat conservation requires.

(v) **Reformulation as defect.** Using $e^3 = 4\omega$, the correction becomes

$$\frac{\omega - h^\vee}{e^3\mu} = \frac{1}{4\mu} \left(1 - \frac{h^\vee}{\omega} \right), \quad (16)$$

so $R^* = 207 \left(1 - \frac{1-h^\vee/\omega}{4\mu} \right)$. The factor $(1 - h^\vee/\omega) = 1 - 12/\pi^3 \approx 0.613$ is the fractional excess of the fundamental frequency over the Coxeter-normalised frequency $h^\vee$. This decomposition suggests a renormalisation interpretation: the “bare” mass ratio $207/1 = 207$ is corrected by the fractional defect of ω relative to its minimal Lie-algebraic anchor $h^\vee$.

4.3 Gap to proof

The following steps are required to elevate Conjecture 4.1 from “supported by evidence” to an established result:

1. **Derivation of the 207 prefactor** from the 4-simplex geometry. P208 presented a candidate derivation; it has not been fully formalised. Without the 207 prefactor being derived, the formula R^* as a whole cannot be proved.
2. **Operator integral giving $(\omega - h^\vee)$.** A computation over the lepton face for instance, a Weyl-character-type integral of the A_2 root system evaluated at the geodesic angle $\theta_4 = \arccos(-\frac{1}{4})$ should reproduce the factor $\omega - h^\vee$. No such computation has been attempted.
3. **Ruling out the triple coincidence.** Since $h^\vee(A_2) = h(A_2) = |\Phi^+(A_2)| = 3$, the conjecture is ambiguous between three Lie-algebraic invariants. A derivation must identify which one is structural. The heat-conservation argument favours $h^\vee$ (via the loop-correction analogy), but this is heuristic.
4. **Connection to P18-T2.** The parent open problem OP-3 asks whether R^* is derivable from the Wheel $\leftrightarrow \hat{O}$ identification. Establishing this would require showing that the A_2 root data of the lepton face follow from the algebraic structure of $\hat{O}$ (Paper 18), not merely from geometric coincidence.

Until items (1)–(4) are resolved, Conjecture 4.1 remains OPEN.

5 Open Problems Update

Open Problem 5.1 (OP-3: P18-T2 and the mass ratio correction integer). **Status: OPEN (conjectured).**

The formula $R^* = 207(1 - (\omega - h^\vee(A_2))/(e^3\mu))$ achieves 1.124 ppm agreement with CODATA. The identification $3 = h^\vee(A_2)$ is supported by:

- The integer scan (unique minimum at $n = 3$; $200\times$ better than $n = 2$).
- The A_2 face type of the lepton triangle (P203).
- The heat-conservation / loop-correction analogy.
- The reformulation $R^* = 207(1 - (1 - h^\vee/\omega)/(4\mu))$ as a “renormalised defect” formula.

Not yet proved: derivation of the 207 prefactor; operator integral reproducing $\omega - h^\vee$; disambiguation of $h^\vee$ vs h vs $|\Phi^+|$; connection to the $\hat{O}$ algebraic structure of Paper 18.

Open Problem 5.2 (OP-3b: The 1.124 ppm residual). **Status: OPEN.**

The gap $|R^* - \text{CODATA}| = 1.124 \text{ ppm}$ is below the $\sim 10 \text{ ppm}$ one-loop QED correction scale but above the $\sim 0.001 \text{ ppm}$ two-loop scale. Its origin is unknown. Candidates: (a) CODATA uncertainty in the input value used (206.768 283 0, 6 significant figures); (b) a genuine two-loop correction involving a second face of the 4-simplex; (c) a correction from the non-lepton vertices v_3, v_4 .

Open Problem 5.3 (OP-3c: A_2 triple coincidence disambiguation). **Status: OPEN.**

For A_2 : $h^\vee = h = |\Phi^+| = 3$. The structural argument cannot distinguish these three invariants using numerical data alone. An independent derivation most likely through the Weyl denominator formula or the quadratic Casimir is needed.

6 Summary

This addendum attempts to resolve Open Problem OP-3 by identifying the integer 3 in the mass ratio correction with the dual Coxeter number $h^\vee(A_2) = 3$ of the root system carried by the triangular faces of the 4-simplex on S^3 .

What was established.

- The integer scan $n = 1, \dots, 10$ has a unique minimum at $n = 3$ (1.124 ppm); the next closest is $n = 2$ at 234 ppm.
- The correction $(\omega - h^\vee)/(e^3\mu)$ admits the exact reformulation $(1 - h^\vee/\omega)/(4\mu)$ using $e^3 = 4\omega$.
- The 3×3 Gram matrix of the lepton face has eigenvalues $\frac{5}{4}$ ($\gg 2$) and $\frac{1}{2}$ ($\gg 1$); its trace equals $V = h^\vee(A_2) = 3$.
- The boundary Euler characteristic $\chi_\partial = V - E = 0$ reflects the balanced (A_2 -equidistant) structure of the lepton face.

What remains open. The identification is a conjecture. Four proof obligations are listed in Section 4. OP-3 is recorded as **CONJECTURED**: an integer scan argument and structural analogy support $h^\vee(A_2)$, but no derivation from first principles has been achieved.