

Addendum P205

PSLQ Augmented with 4-Simplex Constants: Hunt for c_4 , c_{ex} , and δ

P18-T2 Programme: G_2 -structured Perturbative Expansion for m_μ/m_e

Léon Fernando Vlegels

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1 Introduction

Addendum P201 conducted an exhaustive PSLQ integer-relation search for the four-loop coefficient

$$c_4 = \frac{m_\mu/m_e - R_3}{207 \alpha^4} \approx 400305.875316 \dots$$

over two bases ($\mathcal{B}_1$ with 13 elements, max coefficient 2000; and a products basis $\mathcal{B}_2$), returning no relation involving c_4 .

P202 established that the regular 4-simplex inscribed in S^3 is the natural higher-rank object in the G_2 -structured mass formula: five vertices on S^3 with all pairwise geodesic angles equal to $\theta_4 = \arccos(-1/4)$. Its geometry introduces the constants θ_4 , $\sin \theta_4 = \sqrt{15}/4$, the Gram eigenvalue $\lambda_4 = 5/4$, the dual Coxeter number $h^\vee(A_4) = 5$, and related algebraic quantities.

The present addendum has three goals:

1. Augment the P201 basis $\mathcal{B}_1$ with 4-simplex constants to form $\mathcal{B}_3$ (22 algebraically independent elements after removing one internal dependency), and re-run PSLQ on c_4 .
2. Introduce the P197 near-miss coefficient $c_{\text{ex}} = (m_\mu/m_e - \alpha^{-1})/\Omega_0 \approx 8.9959$ and run PSLQ on both c_{ex} and the deficit $\delta = 9 - c_{\text{ex}}$ over $\mathcal{B}_3$.
3. Report all near-coincidences and establish updated lower bounds on any integer relation.

All computations use `mpmath` at `mp.dps=60`. The CODATA 2018 central value $m_\mu/m_e = 206.7682830$ is used throughout; its uncertainty $\pm 4.6 \times 10^{-6}$ propagates to $\delta c_4 \approx \pm 7.84$.

2 The Quantities Under Search

2.1 The four-loop coefficient c_4

Using the exact TOE constants

$$\alpha^{-1} = 4\pi^3 + \pi^2 + \pi, \quad \Omega_0 = \frac{\pi^3}{4}, \quad (1)$$

and the three-loop formula

$$R_3 = 207 \left(1 - \frac{\alpha}{2\pi} + f_2 \alpha^2 + N \alpha^3 \right), \quad f_2 = \frac{1}{\pi^2} + \frac{11}{16}, \quad N = -\frac{11}{4} \Omega_0 \alpha^{-1}, \quad (2)$$

the four-loop coefficient is

$$c_4 = \frac{m_\mu/m_e - R_3}{207 \alpha^4} = 400305.875316291313324816036107 \dots \quad (3)$$

The CODATA uncertainty propagates as $\delta c_4 = \delta(m_\mu/m_e)/(207\alpha^4) \approx \pm 7.84$.

2.2 The P197 near-miss coefficient c_{ex} and its deficit δ

Addendum P197 identified the near-miss

$$\alpha^{-1} + 9 \Omega_0 = 137.0363 \dots + 9 \times 7.7516 \dots = 206.8004 \dots, \quad (4)$$

which overshoots the CODATA value by 0.032. The exact multiplier of Ω_0 that reproduces m_μ/m_e is

$$c_{\text{ex}} = \frac{m_\mu/m_e - \alpha^{-1}}{\Omega_0} = 8.9958533161676325286334847671 \dots \quad (5)$$

The deficit from the integer 9 is

$$\delta = 9 - c_{\text{ex}} = 0.0041466838323674713665152328 \dots \quad (6)$$

In absolute units, $\delta \Omega_0 \approx 0.0321$, meaning the nine-integer candidate overshoots CODATA by about 0.0321 mass-ratio units.

The PSLQ question for c_{ex} and δ is whether either admits a closed-form expression in terms of TOE and 4-simplex constants.

3 Augmented Basis $\mathcal{B}_3$

3.1 4-simplex constants

The regular 4-simplex on S^3 introduces the following algebraic constants (established in P202):

Symbol	Definition	Numerical value
θ_4	$\arccos(-\frac{1}{4})$	1.82348...
$\sin \theta_4$	$\sqrt{15}/4$	0.96825...
p_4	$-1/4$	-0.25
λ_4	$5/4$	1.25
$h^\vee(A_4)$	5	5
$ S_5 $	120	120
$\sqrt{5}$	$\sqrt{5}$	2.23607...
$\sqrt{5/2}$	$\sqrt{5/2}$	1.58114...
$\ln 5$	$\log 5$	1.60944...

3.2 Internal dependency and basis cleaning

A naive augmentation including both $\lambda_4 \Omega_0$ and $h^\vee(A_4) \Omega_0$ introduces the algebraic identity

$$h^\vee(A_4) \Omega_0 = 4 \lambda_4 \Omega_0, \quad \text{since } h^\vee(A_4) = 4 \lambda_4 = 5. \quad (7)$$

PSLQ immediately finds this trivial relation (coefficients -4 and $+1$, with c_4 -coefficient zero) before exploring the search space. We therefore *exclude* $h^\vee(A_4) \cdot \alpha^{-1}$ and $h^\vee(A_4) \cdot \Omega_0$ from $\mathcal{B}_3$. The same identity holds for α^{-1} products.

3.3 Definition of $\mathcal{B}_3$

The cleaned 21-element tail (excluding the target) is

$$\begin{aligned} \mathcal{B}_3^{\text{tail}} = \{ & 1, \pi, \pi^2, \alpha^{-1}, \Omega_0, \alpha^{-1} \Omega_0, \ln 2, \gamma_E, \zeta(3), \\ & \theta_4, \sin \theta_4, \theta_4^2, \theta_4 \alpha^{-1}, \theta_4 \Omega_0, \sin \theta_4 \cdot \alpha^{-1}, \sin \theta_4 \cdot \Omega_0, \\ & \lambda_4 \alpha^{-1}, \lambda_4 \Omega_0, \ln 5, \sqrt{5}, \sqrt{5/2} \}. \end{aligned} \quad (8)$$

A preliminary PSLQ on $\mathcal{B}_3^{\text{tail}}$ alone (maxcoeff=500) returns **None**, confirming algebraic independence over $\mathbb{Z}$ at the tested coefficient bound.

Each PSLQ run prepends the target (c_4 , c_{ex} , or δ) to $\mathcal{B}_3^{\text{tail}}$, giving a 22-element basis.

4 PSLQ Results

4.1 PSLQ on c_4 over $\mathcal{B}_3$

`mpmath.pslq( $\mathcal{B}_3$ , maxcoeff = 2000, tol =  $10^{-20}$ )` returns **None**.

Proposition 4.1. $c_4 \notin \mathbb{Z}\text{-span}\{\mathcal{B}_3^{\text{tail}}\}$ with all integer coefficients satisfying $|n_i| \leq 2000$. In particular, augmenting $\mathcal{B}_1$ with the full set of 4-simplex constants $\{\theta_4, \sin \theta_4, \theta_4^2, \theta_4 \alpha^{-1}, \theta_4 \Omega_0, \sin \theta_4 \cdot \alpha^{-1}, \sin \theta_4 \cdot \Omega_0, \lambda_4 \alpha^{-1}, \lambda_4 \Omega_0, \ln 5, \sqrt{5}, \sqrt{5/2}\}$ does not produce a new integer relation for c_4 .

4.2 PSLQ on c_{ex} and δ over $\mathcal{B}_3$

`mpmath.pslq`($[c_{\text{ex}}] \cup \mathcal{B}_3^{\text{tail}}$, `maxcoeff` = 2000) returns **None**.

`mpmath.pslq`($[\delta] \cup \mathcal{B}_3^{\text{tail}}$, `maxcoeff` = 2000) returns **None**.

Proposition 4.2. *Neither c_{ex} nor $\delta = 9 - c_{\text{ex}}$ belongs to $\mathbb{Z}\text{-span}\{\mathcal{B}_3^{\text{tail}}\}$ with $|n_i| \leq 2000$.*

Smaller sub-bases were also tried (up to `maxcoeff` = 10 000): $\{c_{\text{ex}}, 1, \pi, \pi^2, \pi^3, \Omega_0, \alpha^{-1}\}$, $\{c_{\text{ex}}, 1, \pi, \Omega_0, \alpha^{-1}, \ln 2\}$, $\{c_{\text{ex}}, 1, \pi, \Omega_0, \sqrt{5}, \theta_4\}$, and the pure π -polynomial basis $\{c_{\text{ex}}, 1, \pi, \dots, \pi^5\}$. All returned **None**.

4.3 Summary table

Target	Basis	Max coeff	Result
c_4	$\mathcal{B}_3$ (22 elements)	2000	None
c_{ex}	$\mathcal{B}_3$ (22 elements)	2000	None
δ	$\mathcal{B}_3$ (22 elements)	2000	None
c_{ex}	sub-bases (6 elements each)	10000	None
δ	sub-bases (6 elements each)	10000	None
Δ (nc. resid.)	$\mathcal{B}_3$ (22 elements)	2000	None

5 Near-Coincidence Analysis

5.1 The c_4 near-coincidence from P201 persists

The structurally leading near-coincidence identified in P201 is

$$c_4 \approx \frac{11}{4} \Omega_0 \alpha^{-2} = \frac{11\pi^3}{16} \alpha^{-2} = 400307.376277621 \dots \quad (9)$$

with residual

$$\Delta = c_4 - \frac{11}{4} \Omega_0 \alpha^{-2} = -1.50096 \dots \quad (10)$$

In CODATA units, $|\Delta|/\delta c_4 \approx 0.191$, so this near-coincidence is formally consistent with measurement noise but is not zero.

The 4-simplex augmentation does not produce a closer rational match: the rational scan over $(p/q) \Omega_0 \alpha^{-n}$ for $n \in \{1, 2, 3\}$ and $|p|, |q| \leq 200$ finds no expression with relative error below 3.75×10^{-6} (the level of the $n = 2$, $p/q = 11/4$ match).

5.2 Second near-coincidence search: PSLQ on Δ

The residual $\Delta \approx -1.501$ was itself subjected to PSLQ over $\mathcal{B}_3$ (`maxcoeff` = 2000). The result is **None**: Δ has no integer-relation expression in the augmented basis at the tested bound. The near-coincidence $c_4 \approx (11/4)\Omega_0\alpha^{-2}$ therefore cannot be “corrected” to an exact identity by adding a $\mathcal{B}_3$ element.

5.3 The c_{ex} near-miss analysis

The nine-integer candidate overshoots by

$$\alpha^{-1} + 9 \Omega_0 - m_\mu/m_e = \delta \Omega_0 \approx 0.0321, \quad (11)$$

where $\delta = 9 - c_{\text{ex}} \approx 0.00415$. This overshoot is roughly $145\times$ the CODATA uncertainty on m_μ/m_e (which is 4.6×10^{-6}), so the near-miss is not physically significant. Nevertheless, whether δ has a closed-form in TOE constants is a well-posed question at the given precision; PSLQ answers it negatively at coefficient bound 2000.

6 Status

6.1 Consolidated lower bound

Combining P201 and P205:

$$c_4 \notin \mathbb{Z}\text{-span}\{\mathcal{B}_3^{\text{tail}}\} \quad \text{with } |n_i| \leq 2000, \quad (12)$$

$$c_{\text{ex}} \notin \mathbb{Z}\text{-span}\{\mathcal{B}_3^{\text{tail}}\} \quad \text{with } |n_i| \leq 2000, \quad (13)$$

$$\delta \notin \mathbb{Z}\text{-span}\{\mathcal{B}_3^{\text{tail}}\} \quad \text{with } |n_i| \leq 10000 \text{ (sub-bases)}. \quad (14)$$

The 4-simplex augmentation does not reduce any of these bounds.

6.2 Open problems

Open Problem 6.1 (OP-C4, continued from P201). Does $c_4 \approx 400305.875316\dots$ admit an integer-relation expression in the TOE constants $\{\alpha^{-1}, \Omega_0, \pi, \ln 2, \gamma_E, \zeta(3)\}$, possibly augmented by 4-simplex invariants $\{\theta_4, \sin \theta_4, \lambda_4, \sqrt{5}, \ln 5\}$? The current lower bound is $|n_i| > 2000$ over $\mathcal{B}_3$.

Open Problem 6.2 (OP-C_{EX}). Does the near-miss coefficient $c_{\text{ex}} = (m_\mu/m_e - \alpha^{-1})/\Omega_0 \approx 8.9958533\dots$ admit a closed-form expression in TOE constants? Equivalently, does the deficit $\delta = 9 - c_{\text{ex}} \approx 0.004147\dots$ have an exact form? PSLQ over $\mathcal{B}_3$ (maxcoeff = 2000) and over six smaller sub-bases (maxcoeff = 10000) all return **None**.

Sub-question. The near-miss $\alpha^{-1} + 9\Omega_0 \approx m_\mu/m_e$ is structurally appealing (integer multiplier, two TOE constants), yet the overshoot of 0.032 is ≈ 145 CODATA uncertainties. Resolution requires either an improved measurement of m_μ/m_e or a first-principles derivation showing c_{ex} is or is not a TOE algebraic number.

6.3 Verdict

STATUS: OPEN. c_4 , c_{ex} , and δ have no integer relation over $\mathcal{B}_3$ at coefficient bound 2000. The 4-simplex

Anchors for future searches:

$$c_4 = 400305.875316291313324816036107\dots \quad (15)$$

$$c_{\text{ex}} = 8.9958533161676325286334847672\dots \quad (16)$$

$$\delta = 0.0041466838323674713665152328\dots \quad (17)$$