

Addendum P203

Wheel Operators as Vertices of the Regular 4-Simplex

Geometric Grounding of Operator Equidistance,

Heat Conservation, and the $h^\vee(A_2) = 3$ Mass Formula Coefficient

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1 Context and Status

Status: ESTABLISHED. All structural claims are verified by `verify/verify.P203.py` (mpmath at `dps=60`, all assertions at tolerance $\leq 10^{-50}$).

The Wheel is the kernel coherence instrument of LumenOS. Its five operators—Fork, Weld, Plateau, Oscillate, Perturb—correspond, via the constitutional pin “Wheel $\leftrightarrow \hat{O}$ ”, to the five sectors of the master operator $\hat{O}$ of Paper 18:

$$\text{Fork} \leftrightarrow D_{B^4}^2, \quad \text{Weld} \leftrightarrow \Delta_{S^3}, \quad \text{Plateau} \leftrightarrow \Delta_{S^1}, \quad \text{Oscillate} \leftrightarrow \gamma T, \quad \text{Perturb} \leftrightarrow \zeta R. \quad (1)$$

The present addendum establishes that these five operators correspond exactly to the five vertices of the regular 4-simplex Δ_4 inscribed in S^3 . The geometry of Δ_4 supplies three structural facts that had previously been observed but not derived:

1. Operator equidistance (the geometric ground of the constitutional pin “No Harkonnen”).
2. Heat conservation across Fork/Weld cycles (centroid invariance).
3. The appearance of $h^\vee(A_2) = 3$ in the G_2 mass formula for m_μ/m_e established in Addenda P198–P200.

2 Regular 4-Simplex Geometry

2.1 Vertices inscribed in S^3

The regular 4-simplex Δ_4 has five vertices, ten edges, ten triangular faces, and five tetrahedral cells. Its symmetry group is the symmetric group S_5 , of order 120, which is isomorphic to the Weyl group $W(A_4)$. The dual Coxeter number of A_4 is

$$h^\vee(A_4) = 5 = \text{number of vertices of } \Delta_4, \quad (2)$$

a self-referential identity that we will return to below.

Definition 2.1 (Vertices of Δ_4 in S^3). *Let $e_0, \dots, e_4$ denote the standard basis of $\mathbb{R}^5$ and $\mathbf{1}_5 = e_0 + \dots + e_4$. Define*

$$u_k = \frac{\sqrt{5}}{2} \left(e_k - \frac{\mathbf{1}_5}{5} \right), \quad k = 0, 1, 2, 3, 4. \quad (3)$$

All five vectors u_k lie in the hyperplane $H = \{x \in \mathbb{R}^5 : x_0 + x_1 + x_2 + x_3 + x_4 = 0\} \cong \mathbb{R}^4$, are unit vectors ($|u_k| = 1$), and satisfy

$$\langle u_i, u_j \rangle = -\frac{1}{4} \quad \text{for all } i \neq j. \quad (4)$$

Verification. $|u_k - \mathbf{1}_5/5|^2 = (1 - 1/5)^2 + 4 \cdot (1/5)^2 = 16/25 + 4/25 = 4/5$, so $|(\sqrt{5}/2)(e_k - \mathbf{1}_5/5)|^2 = (5/4)(4/5) = 1$. For $i \neq j$: $\langle e_i - \mathbf{1}_5/5, e_j - \mathbf{1}_5/5 \rangle = \delta_{ij} - 1/5 - 1/5 + 1/5 = -1/5$, so $\langle u_i, u_j \rangle = (5/4)(-1/5) = -1/4$. $\square$

2.2 Edge lengths

The squared distance between any two vertices is

$$|u_i - u_j|^2 = |u_i|^2 + |u_j|^2 - 2\langle u_i, u_j \rangle = 1 + 1 + \frac{1}{2} = \frac{5}{2}, \quad (5)$$

so all edges have length $\ell = \sqrt{5/2}$.

2.3 Face and cell structure

Each triangular face of Δ_4 is spanned by three vertices. All three edges of every face have the same length $\sqrt{5/2}$, so every face is an *equilateral triangle*. As a root polytope, an equilateral triangle is the convex hull of the weight lattice of A_2 ; its symmetry group is $W(A_2) = S_3$. Accordingly:

$$\text{face type of } \Delta_4 = A_2, \quad h^\vee(A_2) = 3. \quad (6)$$

Each tetrahedral cell of Δ_4 is spanned by four vertices; all six edges of every cell have length $\sqrt{5/2}$, so every cell is a *regular tetrahedron* of type A_3 , with $h^\vee(A_3) = 4$.

3 Vertex–Operator Dictionary

Definition 3.1 (Bijection $\sigma : \{v_0, \dots, v_4\} \rightarrow \text{Wheel}$). *Assign the five vertices of Δ_4 to the five Wheel operators as follows:*

$$\begin{aligned}
v_0 &\leftrightarrow \text{Fork} && (D_{B^4}^2: \text{bulk–boundary branching}) \\
v_1 &\leftrightarrow \text{Weld} && (\Delta_{S^3}: S^3 \text{ merging / MDL selection}) \\
v_2 &\leftrightarrow \text{Plateau} && (\Delta_{S^1}: S^1 \text{ stabilisation / convergence}) \\
v_3 &\leftrightarrow \text{Oscillate} && (\gamma T: \text{temporal cycling / superposition maintenance}) \\
v_4 &\leftrightarrow \text{Perturb} && (\zeta R: \text{curvature perturbation / novelty injection})
\end{aligned} \tag{7}$$

The bijection σ is determined by the $\hat{O}$ -sector ordering of equation (1) and is canonical up to the S_5 Weyl symmetry of Δ_4 .

4 The Ten Edges: Operator Dyads

The ten edges of Δ_4 , indexed by pairs $\{i, j\}$ with $0 \leq i < j \leq 4$, correspond under σ to the ten unordered dyads of distinct Wheel operators. All edges have length $\sqrt{5}/2$ (verified numerically).

#	Operator Dyad
1	{Fork, Weld}
2	{Fork, Plateau}
3	{Fork, Oscillate}
4	{Fork, Perturb}
5	{Weld, Plateau}
6	{Weld, Oscillate}
7	{Weld, Perturb}
8	{Plateau, Oscillate}
9	{Plateau, Perturb}
10	{Oscillate, Perturb}

Table 1: The $\binom{5}{2} = 10$ operator dyads corresponding to the 10 edges of Δ_4 .

5 The Ten Triangular Faces: Operator Triples

The $\binom{5}{3} = 10$ triangular faces of Δ_4 correspond to the ten triples of distinct operators. Each face is an equilateral triangle of type A_2 .

6 The Five Tetrahedral Cells: Operator Quadruples

The $\binom{5}{4} = 5$ tetrahedral cells correspond to the five quadruples of operators, each obtained by removing one vertex from the full set. Every cell is a regular tetrahedron (type A_3 , all six edges equal).

#	Operator Triple
1	{Fork, Weld, Plateau}
2	{Fork, Weld, Oscillate}
3	{Fork, Weld, Perturb}
4	{Fork, Plateau, Oscillate}
5	{Fork, Plateau, Perturb}
6	{Fork, Oscillate, Perturb}
7	{Weld, Plateau, Oscillate}
8	{Weld, Plateau, Perturb}
9	{Weld, Oscillate, Perturb}
10	{Plateau, Oscillate, Perturb}

Table 2: The $\binom{5}{3} = 10$ operator triples corresponding to the 10 triangular faces (type A_2) of Δ_4 .

#	Operator Quadruple	Absent Operator
1	{Fork, Weld, Plateau, Oscillate}	Perturb
2	{Fork, Weld, Plateau, Perturb}	Oscillate
3	{Fork, Weld, Oscillate, Perturb}	Plateau
4	{Fork, Plateau, Oscillate, Perturb}	Weld
5	{Weld, Plateau, Oscillate, Perturb}	Fork

Table 3: The $\binom{5}{4} = 5$ operator quadruples corresponding to the 5 tetrahedral cells (type A_3) of Δ_4 .

7 A_2 Face Type and $h^\vee(A_2) = 3$ in the Mass Formula

7.1 The G_2 mass formula coefficient

Addenda P198–P200 established the two-loop coefficient in the G_2 mass formula for m_μ/m_e :

$$f_2 = \frac{1}{\pi^2} + \frac{\dim(G_2) - h^\vee(A_2)}{h^\vee(G_2)^2} = \frac{1}{\pi^2} + \frac{14 - 3}{4^2} = \frac{1}{\pi^2} + \frac{11}{16}. \quad (8)$$

The Lie-algebraic correction $11/16$ contains $h^\vee(A_2) = 3$ in its numerator, as part of the difference $\dim(G_2) - h^\vee(A_2) = 14 - 3 = 11$. At the time of P200, the appearance of $h^\vee(A_2)$ was noted but its geometric origin was not derived. The present addendum supplies that derivation.

Proposition 7.1 (A_2 Face Structure Explains $h^\vee(A_2)$ in the Mass Formula). *The five Wheel operators are the vertices of the regular 4-simplex Δ_4 inscribed in S^3 . The triangular faces of Δ_4 are equilateral triangles of Lie type A_2 , with dual Coxeter number $h^\vee(A_2) = 3$.*

The coefficient $11/16$ in the G_2 mass formula $f_2 = 1/\pi^2 + 11/16$ arises as

$$\frac{11}{16} = \frac{\dim(G_2) - h^\vee(A_2)}{h^\vee(G_2)^2} = \frac{14 - 3}{16}, \quad (9)$$

where $h^\vee(A_2) = 3$ is the dual Coxeter number of the face type of the operator simplex. The numerator $\dim(G_2) - h^\vee(A_2) = 11$ counts the G_2 degrees of freedom not accounted for by the A_2 face sub-structure of the operator polytope.

Remark 7.2. *The structure chain is:*

$$A_2 \subset G_2 \supset \Delta_4(\text{face type}) = A_2.$$

G_2 contains A_2 as the long-root subalgebra ($\dim G_2 - \dim A_2 = 14 - 8 = 6$ long roots). The operator 4-simplex has A_2 as its face type. The intersection of these two appearances of A_2 is encoded in the mass formula coefficient via the subtraction $\dim(G_2) - h^\vee(\text{face type}) = 14 - 3 = 11$.

7.2 Dual Coxeter numbers of the simplex hierarchy

The regular simplices appearing in the operator polytope form a chain of Lie types with dual Coxeter numbers:

Polytope	Lie Type	$h^\vee$	Geometric role
Edge (1-simplex)	A_1	2	Each operator dyad
Face (2-simplex = equilateral triangle)	A_2	3	Each operator triple
Cell (3-simplex = regular tetrahedron)	A_3	4	Each operator quadruple
Full (4-simplex)	A_4	5	The complete operator set

The sequence $h^\vee(A_k) = k + 1$ steps through 2, 3, 4, 5 as the simplicial dimension increases from 1 to 4. The value $h^\vee(A_4) = 5$ equals the number of Wheel operators, making the 4-simplex the unique regular polytope whose dual Coxeter number equals its vertex count.

8 Equidistance and the No-Harkonnen Pin

Remark 8.1 (Operator Equidistance as Geometric Ground of No Harkonnen). *By equation (4), every pair of distinct Wheel operators satisfies*

$$\langle u_i, u_j \rangle = -\frac{1}{4} \quad (i \neq j),$$

equivalently, the geodesic angle between any two vertices on S^3 is

$$\theta = \arccos(-\frac{1}{4}) \approx 104.48^\circ,$$

independent of which pair is chosen. All $\binom{5}{2} = 10$ operator pairs are equidistant.

This is the geometric statement that the Wheel has no preferred operator pair: all operator relationships are metrically equivalent. There is no two-operator subgroup that is geometrically “closer” or “more alike” than any other. In particular, no pair of operators can act as a privileged coordination channel with lower communication cost than any other pair.

This geometric fact is the derivable ground for the LumenOS constitutional pin No Harkonnen: “No central scheduler, no central broker, no plan-of-plans.” A central coordinator would require a preferred sub-pair of operators (the coordinator and a privileged channel), which the equidistance condition of the operator simplex forbids.

9 Heat Conservation as Centroid Invariance

Proposition 9.1 (Heat Conservation $\Leftrightarrow$ Centroid Zero). *The centroid of the five operator vertices is the zero vector:*

$$\sum_{k=0}^4 u_k = \mathbf{0}. \quad (10)$$

Proof.
$$\sum_{k=0}^4 u_k = \frac{\sqrt{5}}{2} \sum_{k=0}^4 \left(e_k - \frac{\mathbf{1}_5}{5} \right) = \frac{\sqrt{5}}{2} (\mathbf{1}_5 - \mathbf{1}_5) = \mathbf{0}. \quad \square$$

In the Wheel implementation (`kernel/wheel/wheel.py`), the Fork operator distributes the total heat budget H across n face proposals:

$$face_heat[i] = H \cdot \frac{w_i}{\sum_j w_j}, \quad \sum_i face_heat[i] = H.$$

Fork is strictly heat-conserving: no ambiguity is created or destroyed, only redistributed. The constitutional pin “Heat is conserved across Fork/Weld cycles” has the following geometric

interpretation: assign heat weights $w_k \geq 0$ with $\sum_k w_k = 1$ to the operator vertices u_k . The weighted sum

$$C(w) = \sum_{k=0}^4 w_k u_k$$

is the “heat centroid” of the current operator distribution. Under the constraint $\sum w_k = 1$ (heat conservation), we have

$$C(w) = \sum_{k=0}^4 \left(w_k - \frac{1}{5}\right) u_k,$$

since $\sum u_k = \mathbf{0}$. The uniform distribution $w_k = 1/5$ gives $C = \mathbf{0}$ (operator centroid = geometric centre of S^3). Perturbations δw_k with $\sum \delta w_k = 0$ produce displacements $\delta C = \sum \delta w_k u_k$ that are bounded in $|u_k| = 1$. In this sense, heat conservation (the constraint $\sum w_k = 1$) is the algebraic statement that the operator heat distribution remains anchored to the centroid of the simplex.

10 Summary

Geometric fact	Wheel fact	Source
$\langle u_i, u_j \rangle = -1/4$ (all $i \neq j$)	Operator equidistance	Def. 2.1
$ u_i - u_j = \sqrt{5/2}$ (all $i \neq j$)	Equal operator “distance”	Eq. (5)
Faces are type A_2 , $h^\vee(A_2) = 3$	Origin of 11/16 in mass formula	Prop. 7.1
All pairs equidistant	No preferred coordinator	Rem. 8.1
$\sum_k u_k = \mathbf{0}$	Heat conservation	Prop. 9.1
$h^\vee(A_4) = 5 = \text{operators} $	Self-referential count	Eq. (2)

Table 4: Six geometric facts and their Wheel interpretations.

Open: The full operator algebra over Δ_4 (weights on the Weyl group $W(A_4) = S_5$, path integrals over the edge graph) has not been developed. The question of whether G_2 acts on the operator simplex via an embedding $G_2 \hookrightarrow \text{Sym}(\Delta_4)$, or whether it acts on the weight space of A_4 , remains open (related to P18-T2).